Stability and Chaos in a Yang-Mills System

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The question of chaos in the equations of Yang-Mills is not a trivial and straightforward one. We study the stability and chaotic behaviour of a quantum version of a Yang-Mills gauge theory. We show that the Quantification of the theory does not remove the high sensitivity to initial conditions seen in classical case and the non-integrability remains. Then we modelled the interference as a white noise and wrote the equations in the form of a stochastic system of equations (SDE). We demonstrate that the solution of this system is unique and does not explode in a finite time.

1. Introduction

The great achievements of theoretical physics in the 20th century, Einstein's theory of general relativity (G-R) and the Yang-Mills theory (Y-M) of non-Abelian gauge fields, have many common properties: both are gauge theories and they are nonlinear. From this point of view, the chaoticity of the corresponding fields is not a surprise. On the other hand, there are many examples of stable solutions of nonlinear field equations. Thus, the question of chaos in the equations of G-R and Y-M is not a trivial and straightforward one.

The meaning of chaos in a field theoretical system has already been examined. In the context of particle physics, Matinyan et al [1] were the first to show that classical Yang-Mills system is a K-one. Nikolaevsky and Shchur conjectured that if chaos is present in the dynamics of homogeneous field then it is present in the full field theory [2]. This was confirmed in the Y-M field [3]. In this work, we will consider a little bit complication: we add quantum to the previous study.

Chaos is a manifestation of highly sensitive trajectories to initial conditions [4-5]. In quantum physics, there are no trajectories in the classical sense and so there is a problem in connecting dynamical behaviour (chaoticity) with quantum phenomena (both Schrodinger and Dirac quantum evolution equations are linear). We overcome the difficulty by defining chaos as a manifestation of highly sensitive systems to initial conditions or:

"*Big changes in final states induced by small perturbations in initial conditions*"

This allows us to go beyond the classical notion of trajectories to consider only the evolution of states of the quantum system. This is the main property used in our work to study the dynamical behavior in quantum version of gauge theories like Yang-Mills types.

We have shown that the equations of motions are the same as those of the classical system if we neglect the interference term. By using the Painlevé test [6], [7], we demonstrate that the theory is non-integrable. We use also the graphical procedure to make evident the sensitivity of the theory to initial conditions. Finally, we modelled the interference as a white noise and so wrote the equations in the form of a stochastic system of equations. We used the Khasminskii procedure [8] to demonstrate that the solution of this system is unique and does not explode in finite time.

2. Quantum Yang-Mills System as a Dynamical System

We write the Lagrangian of a $SU(2)$ quantum Y-M system as:

$$L = -\frac{1}{4}\left(F_{\mu\nu}^{a}\right)^{2} - \frac{1}{2\xi}\left(\partial_{\mu}A^{\mu}\right)^{2} - \bar{C}^{a}\partial_{\mu}D^{\mu}C^{a} \quad .(1)$$

Latin alphabet $a = 1,2,3$ corresponds to Isospin and Greek symbols $\mu, \nu = 0,1,2,3$ to Lorentz coordinates.

$$F_{\mu\nu}^{a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + g\varepsilon^{abc}A_{\mu}^{b}A_{\nu}^{c} \quad .(2)$$

Where, A_{μ}^{a} are gauge fields and $(\bar{C}^{a})C^{a}$ are anti-ghosts. We have to remark that, in our case, all the fields are operators and thus do not commute.

We use the BRST (Becchi, Rouet, Stora and Tyutin) transformations [9-10] (λ is a Grassmann constant):

$$\begin{cases} \delta A_\mu^a = -\frac{1}{g}\left(D_\mu A^{a\mu}\right)\lambda \\ \delta C^a = -\frac{1}{2}\varepsilon^{abc}C^b C^c\lambda \\ \delta \bar{C}^a = -\frac{1}{g\xi}\left(D_\mu A^{a\mu}\right)\lambda \end{cases} \quad .(3)$$

The quantum equations of motion are as follows:

$$\begin{cases} \partial_\mu F^{a\mu\nu} + g\varepsilon^{abc}A_\mu^b F^{c\mu\nu} + \\ \frac{1}{\xi}\partial^\mu\partial_\mu A^{a\nu} = g\varepsilon^{abc}\bar{C}^b C^a \\ \partial^\mu\left(\partial_\mu C^a + \varepsilon^{abc}A_\mu^b C^c\right) = 0 \\ \partial_\mu\partial^\mu \bar{C}^a + \varepsilon^{ab}\ A_\mu^b\partial^\mu \bar{C}^c = 0 \end{cases} \quad .(4)$$

We use the temporal gauge $A_0^a(x,t) = 0$. The BRST are preserved, if we take: $\partial_0 C^a = 0$; $\partial_0 \bar{C}^a = 0$.

We consider the spatially homogeneous fields:

$$\begin{cases} A_i^a \equiv A_i^a(t) \\ \partial_i A_i^a = 0 \\ \partial_i C_i = 0 \end{cases} \quad .(5)$$

To obtain the new equation of motion

$$\begin{cases} \ddot{A}_i^a + g^2 A_j^b\left(A_i^a A_j^b - A_j^a A_i^b\right) = 0 \\ \prod C^a = 0 \\ \prod \bar{C}^a = 0 \end{cases} \quad .(6)$$

If one makes the substitution (we do not sum over a)

$$A_i^a(t) = O_i^a f^a(t)\ \&\ O_i^a O_i^b = g^{-1} \quad .(7)$$

Where, O_i^a are orthogonal matrices, we can see that the Gauss law is verified

$$\dot{A}_i^a A_j^b - A_i^a \dot{A}_j^b = 0 \quad .(8)$$

The equation of motion takes now the form

$$\ddot{f}^a + \sum_{b\neq a}(f^b)^2 f^a = 0 \quad .(9)$$

We choose a global gauge $O_i^a = \delta_i^a/g$, so Isospin directions become spatial ones $A_i^a = f^a$; equation of motion are

$$\begin{cases} \ddot{f}^1 + (f^2)^2 f^1 = 0 \\ \ddot{f}^2 + (f^1)^2 f^2 = 0 \end{cases} \quad .(10)$$

Where f^1 and f^2 are operators in the Hilbert space. The system is in the physical state, $|\psi_p\rangle \in H$, and the equation of motion can be written in Heisenberg representation as follows (Summation is over q and q'):

$$\begin{cases} \langle\psi_p|f^2\sum|\psi_{q'}\rangle\langle\psi_{q'}|f^2\sum|\psi_q\rangle\langle\psi_q|f^1|\psi_p\rangle \\ \quad +\langle\psi_p|\ddot{f}^1|\psi_p\rangle = 0 \\ \langle\psi_p|f^1\sum|\psi_{q'}\rangle\langle\psi_{q'}|f^1\sum|\psi_q\rangle\langle\psi_q|f^2|\psi_p\rangle \\ \quad +\langle\psi_p|\ddot{f}^2|\psi_p\rangle = 0 \end{cases} \quad .(11)$$

Finally, we write it as a dynamical system

$$\begin{cases} \ddot{U} + V^2 U = \left\langle\psi_p\left|\begin{matrix} f^2\sum_{q'\neq p}|\psi_{q'}\rangle\langle\psi_{q'}| \\ f^2\sum_{q\neq p}|\psi_q\rangle\langle\psi_q|f^1 \end{matrix}\right|\psi_p\right\rangle \\ \ddot{V} + U^2 V = \left\langle\psi_p\left|\begin{matrix} f^1\sum_{q'\neq p}|\psi_{q'}\rangle\langle\psi_{q'}| \\ f^1\sum_{q\neq p}|\psi_q\rangle\langle\psi_q|f^2 \end{matrix}\right|\psi_p\right\rangle \end{cases} \quad .(12)$$

Taking,

$$\begin{cases} U = \langle\psi_p|f^1|\psi_p\rangle \\ V = \langle\psi_p|f^2|\psi_p\rangle \end{cases} \quad .(13)$$

Our quantization procedure differs from the one used by Nicolaidis et al. in [12], but we consider that it is more direct and more consistent with the principles of quantum mechanics.

The right terms in equation (12) are interference parts. We start by taking them equal to zero by considering that there are no connections between $|\psi_p\rangle$ and the other states, and then we will model them as a white noise.

3. Painlevé Test

If we consider that there is no connections between the physical state $|\psi_p\rangle$ and the other states $|\psi_p\rangle$, then equations (12) becomes identical to the classical ones [1]

$$\begin{cases} \ddot{U} + V^2 U = 0 \\ \ddot{V} + U^2 V = 0 \end{cases} \quad .(14)$$

This system corresponds to the case of the quartic potential x^2y^2 that has been widely studied for its strong chaotic behaviour and its applications in different domains both in physics and chemistry (for further information, one can see [13 -16] and references therein).

We mention here that, unlike the aforementioned references, the elements in (10) are operators and not real functions, and this is what led us to use the mean values in the state of the system to get (11) and (12). We start studying the chaoticity of this system using the Painlevé test [6-7] based on three steps:

1- Determine if the solutions are analytically continued;
2- Determine the leading singularity;
3- If it is not more than a branch or a pole, then make Laurent expansion. Determine the power for which the coefficients become arbitrary, Kowalewsky exponents or resonances.

Then one can use the necessary condition for the system to be integrable.

4. Kowalewsky exponents should be rational numbers

Let us pass the test now. To find the leading singularity, we make the ansatz:

$$\begin{cases} U(t) = a(t-t_0)^{-\alpha} \\ V(t) = b(t-t_0)^{-\beta} \end{cases} \quad .(15)$$

Where, t_0 represents the movable singularity location.

Replacing in the equations and balancing the most singular terms give us the following:

$$\begin{cases} a\alpha(\alpha+1)(t-t_0)^{-\alpha-2} \\ +ab^2(t-t_0)^{-\alpha-2\beta}=0 \\ b\beta(\beta+1)(t-t_0)^{-\beta-2} \\ +a^2b(t-t_0)^{-\beta-2\alpha}=0 \end{cases} \quad .(16)$$

We get

$$\begin{cases} -\alpha-2=-\alpha-2\beta \Longrightarrow \beta=1 \\ a\alpha(\alpha+1)+ab^2=0 \Longrightarrow b^2=-2 \\ -\beta-2=-\beta-2\alpha \Longrightarrow \alpha=1 \\ b\beta(\beta+1)+a^2b=0 \Longrightarrow a^2=-2 \end{cases} \quad .(17)$$

It is OK! We write now the resonances

$$\begin{cases} U=a(t-t_0)^{-\alpha}+p(t-t_0)^{-\alpha+r} \\ V=b(t-t_0)^{-\beta}+q(t-t_0)^{-\beta+r} \end{cases} \quad .(18)$$

Replacing in the equations and balancing terms linear in p and q, we find that

$$(r^2-3r-4)(r^2-3r+4)=0 \quad .(19)$$

Resolving this equation gives us the possible values of r as

$$\begin{cases} r=-1 \\ r=4 \\ r=\left(3+i\sqrt{7}\right)/2 \\ r=\left(3-i\sqrt{7}\right)/2 \end{cases} \quad .(20)$$

The resonances are not rationales (even not real) and our dynamical system fails Painlevé test.

5. Graphical Study

Now we study of the equations (14) using a graphical procedure based on the main characteristic of chaos which is the dependence of the graphs to infinitesimal changes in initials conditions. As an example, we put all the initial conditions equal to 1 (in no way this diminishes the generality of the results).We will focus on the curves representing the solutions of the dynamical system and the representation of solutions in phase space for each variable.

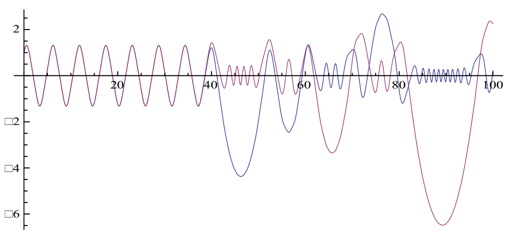

Fig.1. Solutions $U(t)$& $V(t)$ for $U(0)=1, V(0)=1, \dot{U}(0)=1, \dot{V}(0)=1$

We note that the solutions are harmonic up to $t=40s$ and then begins a phase in which they become unpredictable; we call this edge horizon time because it is the time for which the movement is regular and so can be predicted. The same thing can be seen in the phase space of the first variable (Figure 2).

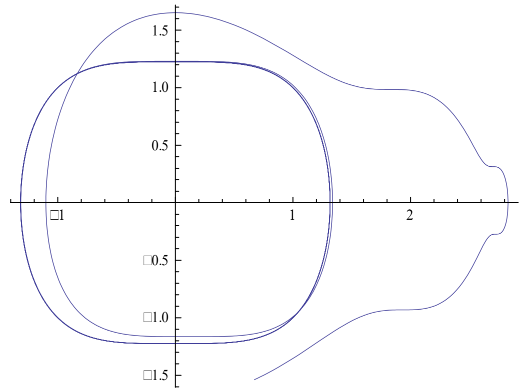

Fig.2. Solution in $(U,\dot{U})$ space for $U(0)=1, V(0)=1, \dot{U}(0)=1, \dot{V}(0)=1$

The closed curve corresponds to the horizon time and characterizes the harmonic motion, while the irregular curve indicates the phase of predictability loss.

In Figure 3, the same behavior is shown in the phase space of the second variable and thus, it is proved that this evolvement to unpredictability is common to the two dynamical variables that define the evolution of our system.

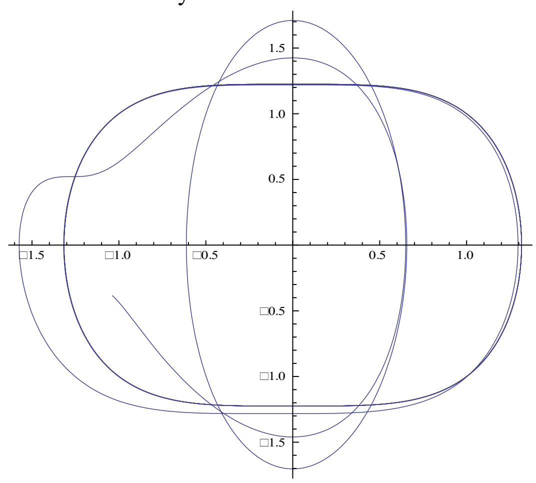

Fig.3. Solution in $(V,\dot{V})$ space for $U(0)=1, V(0)=1, \dot{U}(0)=1, \dot{V}(0)=1$

Now we make a tiny change in initial conditions and see the results on the solutions of our dynamical system. We choose to add 10^{-7} to the value of $U(0)$then to $\dot{U}(0)$and look in each case to changes induced. We chose this value because of Lorenz use a change of 10^{-6} in initial values to demonstrate the presence of chaos in his equations of movement of the air [4-5]. Curves induced by these changes and drawn below and should be compared to the previous ones represented above.

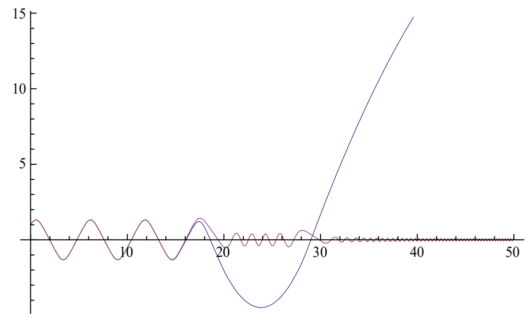

Fig.4. Solutions $U(t)$& $V(t)$ for $U(0) = 1, V(0) = 1 + 10^{-7}, \dot{U}(0) = 1, \dot{V}(0) = 1$

When comparing with Fig.1, one can easily see that the fact of varying, even minimally, just one initial condition strongly changes the shape of the curves. Similarly, the horizon time has also decreased and we can say that if time exceeds this value then it can be regarded as large or infinite compared to the conditions of the experiment. This clearly demonstrates the sensitivity of the final states of the system to the slightest variations in its initial conditions.

The same thing occurs when varying the initial velocity instead of the position (to be compared to Figure 1).

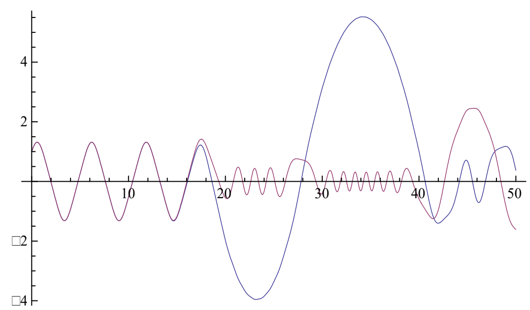

Fig.5. Solutions $U(t)$& $V(t)$ for $U(0) = 1, V(0) = 1, \dot{U}(0) = 1, \dot{V}(0) = 1 + 10^{-7}$

In this case also, the same kinds of variations in the graphs are seen when changing the initial conditions of the first variable with the same amount.

Also in phase spaces, we can see the variations induced by these changes in initial conditions.

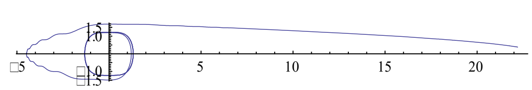

Fig.6. Solution in $(U, \dot{U})$ space for $U(0) = 1, V(0) = 1 + 10^{-7}, \dot{U}(0) = 1, \dot{V}(0) = 1$

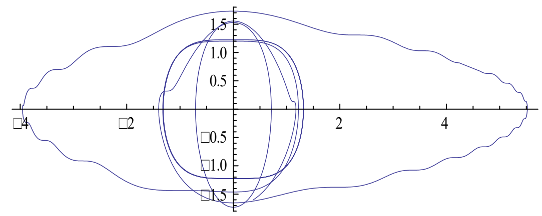

Fig.7: Solution in $(U, \dot{U})$ space for $U(0) = 1, V(0) = 1, \dot{U}(0) = 1, \dot{V}(0) = 1 + 10^{-7}$

Figures 6 and 7 should be compared to Figure 2. The closed curve characterizes the original harmonic motion, but the lines that indicate the irregularities are still present and are also more marked as one can see from the U-axe, where the curves do not exceed the position $U = 3$ in Figure 2, while they pass beyond $U = 4$ in Figure 7 and even go up to 23 in Figure 6.

In the same manner, we show the same effects on curves in the $(V, \dot{V})$ phase space when the results obtained by varying initial conditions are compared to Figure 3.

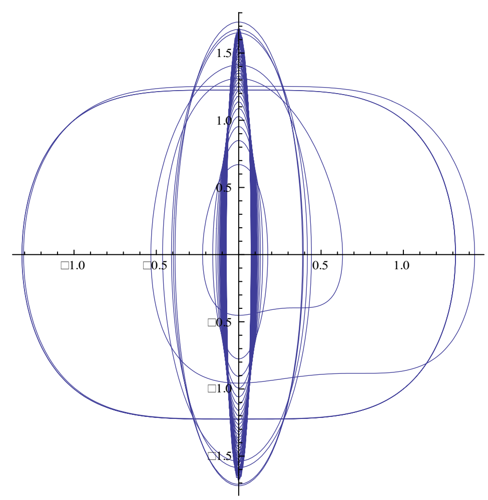

Fig.8. Solution in $(V, \dot{V})$ space for $U(0) = 1, V(0) = 1 + 10^{-7}, \dot{U}(0) = 1, \dot{V}(0) = 1$

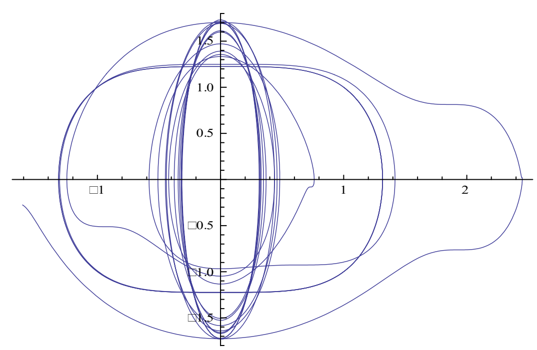

Fig.9. Solution in $(V,\dot{V})$ space for $U(0) = 1, V(0) = 1, \dot{U}(0) = 1, \dot{V}(0) = 1 + 10^{-7}$

6. Stochastic Model:

We write the interferences terms in Eqn. (12) as white noises $\dot{W}$ (a derivative of a Brownian noise W)

$$\begin{cases} \ddot{U} + V^2 U = \dot{W}_1(t) \\ \ddot{V} + U^2 V = \dot{W}_2(t) \end{cases} \quad .(21)$$

This is because we consider quantum mechanics as a random process [17-18-19]. We choose white noise because we have no information on the interferences terms in our system and this noise is a realization of a random process in which the power spectral density is the same for all frequencies [20].

We have also found recently that there are works, which began to investigate potentials consisting of noises in quantum systems [21-23]

We use the notations

$$X(t) = \begin{pmatrix} U(t) \\ V(t) \end{pmatrix}, \sigma(X(t)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b(X(t)) = \begin{pmatrix} V^2(t)U(t) \\ U^2(t)V(t) \end{pmatrix} \quad .(22)$$

Here $X(t), b$ and σ are functions: $X(t): \mathbb{R} \to \mathbb{R}^n, b: \mathbb{R}^n \to \mathbb{R}^n; \sigma: \mathbb{R}^n \to M_{\mathbb{R}}(n)$ where $\mathbb{R}$ are real numbers, $M_{\mathbb{R}}(n)$ real matrices of rank n ($n = 2$ in our case) and time t is always positive.

The equation of motion becomes a stochastic equation

$$\ddot{X}(t) + b(X(t)) = \sigma\big(X(t)\big)\dot{W}(t) \quad .(23)$$

The conditions for a stochastic differential equation to have a unique solution that does not explode in a finite time [8] are

1. $\forall R > 0, \exists K_R > 0 / |b(Y) - b(X)| + |\sigma(Y) - \sigma(X)| \leq K_R |Y - X|, \forall |X| \leq R, \forall |Y| \leq R$
2. X_0 $independant\ of$ $\{W(t), \forall\, t > 0\}$ and $E(X_0) < \infty$
3. $\exists F: \mathbb{R}^n \to \mathbb{R}_+ / \forall X \in \mathbb{R}^n, LF(X) \leq cF(x) \,\&\, F(X) \to +\infty\ if\ |X| \to +\infty$

$E(X_0)$ is the expected value for the random variable X, X_0 is its initial value, c is a positive constant, F is the Lyapounov function (in our case it is a C^2 class one) and L is the differential generator associated with the diffusion process solution of stochastic differential equation

$$L = \sum_{j=1}^{n} b_j(X) \frac{\partial}{\partial X_j} + \frac{1}{2} \sum_{j,k=1}^{n} a_{jk}(X) \frac{\partial^2}{\partial X_j \partial X_k}; a(X) = \sigma(X)\sigma^T(X) \quad .(24)$$

Putting $\dot{Y}(t) = X(t)$ in above equation, we get the stochastic system:

$$\begin{cases} dX(t) = Y(t)dt \\ dY(t) = -b\big(X(t)\big)dt + \sigma\big(X(t)\big)dW(t) \\ X(t=0) = X_0, Y(t=0) = Y_0 \end{cases} \quad .(25)$$

The generator of this system is

$$L = \langle Y, \nabla_X \rangle - \langle b, \nabla_Y \rangle + \frac{1}{2} tr[\sigma(X)\sigma^T(X) D^2 Y] \quad .(26)$$

Where$\langle\ ,\ \rangle$ refers to the scalar product in $\mathbb{R}^2$ and the derivative operators are:

$$\nabla_X = \left(\frac{\partial}{\partial U}, \frac{\partial}{\partial V}\right), \nabla_Y = \left(\frac{\partial}{\partial \dot{U}}, \frac{\partial}{\partial \dot{V}}\right), D^2 Y = \begin{pmatrix} \frac{\partial^2}{\partial U \partial \dot{U}} & \frac{\partial^2}{\partial U \partial \dot{V}} \\ \frac{\partial^2}{\partial V \partial \dot{U}} & \frac{\partial^2}{\partial V \partial \dot{V}} \end{pmatrix} \quad .(27)$$

Now we apply the assumptions for the Khasminskii criterion by finding the F function that fulfills the three conditions. We choose $F = \langle X, Y \rangle / 2 + K$ where K is a positive constant to be sufficiently large to ensure that:

$$F(X,Y) \geq 0, \forall (X,Y) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\&\, F(X,Y) \to 0\ if\ R = |X|^2 + |Y|^2 \to +\infty \quad .(28)$$

We recall that $\dot{Y}(t) = X(t)$ and $F(X,Y)$ is actually a function of a single variable $F(X)$.

The action of the generator L on this function gives:

$$LF(X,Y) = \langle Y, 0 \rangle - \langle b(X), \mathrm{Y} \rangle + 1 \quad .(29)$$

To satisfy the first Khasminskii requirement one has to choose b locally Lipschitz continuous and this ensures that:

$$\exists K_1 > 0, \exists c > 0 \,/\, \langle b(X), Y \rangle + \frac{c|Y|^2}{2} + K_1 \geq 1 \quad .(30)$$

The second Khasminskii requirement is automatically satisfied as the initial conditions in Eqn. 25 are not dependent on the noise.

Now to check the third Khasminskii requirement, we apply the differential generator L on the selected function:

$$LF(X,Y) \leq \frac{c|Y|^2}{2} + K_1 = cF(X,Y) - cK + K_1 \quad .(31)$$

We choose $K \geq \frac{K_1}{c}$ so we get:

$$\forall (X,Y) \in \mathbb{R}^2 \times \mathbb{R}^2, LF(X,Y) \leq cF(X,Y), c > 0 \quad .(32)$$

This last equation achieves the verification of all the Khasminskii conditions for our stochastic differential system and we can conclude that this system, which represent the quantum case, has a unique solution that does not explode in a finite time and we find:

$$Y(t) = \int_0^t b\big(X(s)\big)ds + W(t) - W(0) + Y_0 \quad .(33)$$

This completes our demonstration on the Khasminskii criteria.

7. Conclusion

In this work, we have studied a quantum version of a $SU(2)$ Yang-Mills system whose classical version has been studied from a dynamical point of view. We found that the quantum equations of motion differ from classical ones with a term of interference between different states in the Hilbert space. This is evident because the interference is a major feature of quantum physics [18-19].

We first studied the equations without the interference term and we have demonstrated that it is not integrable in the sense that it has a chaotic behaviour. This was demonstrated by applying the Painlevé test to these equations representing our dynamical system and by studying the variations induced in final states in the graph of solutions of our system by tiny changes in initial conditions, which is the definition of chaos. These noticeable changes in the final states are present even when using variations in initial conditions equal to 10^{-15}.

Then we modelled interferences as white noise, which is consistent with their chaotic nature [18-19]. We investigated the stochastic system obtained using the criteria of Khasminski and demonstrated that the system has a solution that does not explode in a finite time; It may considered here that the interferences act as a regulator of the chaoticity of the original system. But, we must say that this cannot be regarded as an evidence and it will be better if one perform Painlevé conjecture to cover operators.

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Received: 27 February, 2019
Accepted: 13 September, 2019