A Model of Modified Klein–Gordon Equation with Modified Scalar-vector Yukawa Potential

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Here we study the modified Klein-Gordon equation (MKGE) with a unequal mixture of scalar and vector Yukawa (MUSVY) potentials describing the 3-dimensional dynamics in non-commutative quantum mechanics (NCQM) using the Bopp's shift method and standard perturbation theory. The new energy of n^{th} excited state $E^{y}_{r-nc}(a, S_0, V_0, n, j, l, m)$ as a functions of the shift energy $\Delta E(n, j, l, s, m)$ and E_n of an unequal mixture of scalar and vector modified Yukawa potentials is obtained via first-order perturbation theory in the relativistic 3-dimensional non-commutative real space (RNC: 3D-RS) of symmetries. We found that the perturbative solutions of the discrete spectrum dependent on the Gamma function, the discreet atomic quantum numbers (j, l, s, m) and the potential parameters (a, V_0 and S_0), in addition to non-commutativity parameters (θ and σ), which are induced by the effect of (space-space) non-commutative properties. We have also applied our results for a bosonic particle with spin one and have shown that the MKGE under MUSVY potentials become similar to the Duffin–Kemmer equation. In addition, we have shown that the doubled total degeneracy of energy level for bosonic particles with spin one in RNCQM symmetries under the MUSVY potentials is clear physical indicator that physical treatment with RNCQM appear more detailed and clear if compared with similar energy levels obtained in ordinary relativistic quantum mechanics. Furthermore, we considered the nonrelativistic limits of MUSVY potentials.

1. Introduction

The Yukawa potential, also known as static screened Coulomb, Debye-Hückel or Thomas-Fermi potential, of various types has received a great deal of attention in many fields of physics such as nuclear physics, atomic physics, solid-state physics, and astrophysics. This potential has received considerable attention since the early days of quantum mechanics because of its wide range of applications and one of the oldest known since 1935 by researchers and interesting at the microscopic scale [1-4].

Furthermore, this potential also plays a vital role in plasma physics where it is known as the Debye-Huckel potential. In addition to these obvious physical applications, this together with Hulthen and the exponential potentials plays an important role as a good test case in potential scattering studies [4-7]. It is often used to compute bound-state normalizations and energy levels of neutral atoms [8-12]. It also used to compute bound-state normalizations and energy levels of neutral atoms used in (dusty/complex) plasma and colloidal suspensions [13]. Yukawa was the first to propose it and was studied in both relativistic and non-relativistic quantum mechanics [1].

The study of the Yukawa potential in the relativistic Dirac equation in previous years received great attention from many authors [13-15]. They have theoretically and numerically obtained the energy eigenvalues of the system. In [16], Hamzavi and Ikhdair investigated a relativistic quantum mechanical system with the Duffin–Kemmer–Petiau equation for a vector Yukawa potential with arbitrary total angular momenta and obtained the energy eigenvalues of the system. In 2013, M. Hamzavi *et al*. investigated the spin-zero Klein–Gordon particles in the field of a unequal mixture of scalar and vector Yukawa potentials within the framework of the approximation scheme to the centrifugal potential term for any arbitrary *l*-state [17]. In 2016, we have studied this potential in the case of relativistic non-commutative quantum mechanics using the parameters of Bopp's shift method for one-electron atoms with spin half [18]. In particular, Yukawa potential used to describe the interactions of hydrogen-like atoms and neutral atoms. In this work, motivated by several recent studies such as the non-renormalizability of the standard model, string theory, quantum gravity, the non-commutative quantum mechanics NCQM has attracted much attention [19-23]. Furthermore, research findings show that the development of matrix theory and D-branes are achieved in the framework of the symmetries of non-commutative quantum mechanics [24-25].

The idea of non-commutative phase-space was known firstly by Heisenberg in 1930 and was formalized by Snyder in 1947 because of the need to regularize the divergence of the quantum field theory. The aim of the present work is to extend the KG equation in an arbitrary 3-dimension with the unequal mixture of scalar and vector Yukawa potentials to the case of the modified Klein-Gordon equation MKG equation in the non-commutative 3-dimensional space with the unequal modified mixture of scalar and vector Yukawa (MUSVY) potentials in order to find other applications and more profound interpretations in the sub-atomic scales. The non-relativistic energy levels under unequal modified mixture of scalar and vector Yukawa potentials have not been obtained yet in the context of the non-commutative phase-space. Thus, the main purpose of this paper is to solve the MKG with MUSVY potentials in (RNC: 3D-RSP) symmetries (see below):

$$V(r)=-\frac{V_0\exp(-ar)}{r}\rightarrow V(\hat{r})=V(r)-V_0\left(\frac{\exp(-ar)}{2r^3}+\frac{a\exp(-ar)}{2r^2}\right)\vec{\mathbf{L}}\vec{\Theta} \quad (1)$$
$$S(r)=-\frac{S_0\exp(-ar)}{r}\rightarrow S(\hat{r})=S(r)-S_0\left(\frac{\exp(-ar)}{2r^3}+\frac{a\exp(-ar)}{2r^2}\right)\vec{\mathbf{L}}\vec{\Theta}$$

where $V_0=\alpha Z$ (α and Z are the fine-structure constant and the atomic number) while a is the screening parameter. The new structure of RNCQM based to new time-independent NC canonical commutations relations in Schrödinger, Heisenberg and Interactions pictures (SP, HP and IP), respectively, as follows (throughout this paper, the natural units $c=\hbar=1$ will be used) [26-34]:

$$\begin{cases}\left[\hat{x}_i \overset{*}{,} \hat{p}_j\right]=\left[\hat{x}_i(t)\overset{*}{,}\hat{p}_j(t)\right]=\left[\hat{x}_{Ii}(t)\overset{*}{,}\hat{p}_{Ij}(t)\right]=i\hbar_{eff}\delta_{ij}\\ \left[\hat{x}_i \overset{*}{,} \hat{x}_j\right]=\left[\hat{x}_i(t)\overset{*}{,}\hat{x}_j(t)\right]=\left[\hat{x}_{Ii}(t)\overset{*}{,}\hat{x}_{Ij}(t)\right]=i\theta_{ij}\end{cases} \quad (2)$$

Where $\hbar_{eff}$ is the effective Planck constant, $\theta^{ij}=\varepsilon^{ij}\theta$ with θ as the non-commutative parameter, a very small parameter compared to the energy and elements of antisymmetric 3 × 3 real matrix and δ_{ij} is the identity matrix, whereas $(*)$ denote to the Weyl Moyal star product, which is generalized between two arbitrary functions $(f,g)(x)$ to the new form $\hat{f}(\hat{x})\hat{g}(\hat{x})\equiv(f*g)(x)$ in (NC: 3D-RS) symmetries as follows [35-49]:

$$\begin{aligned}(fg)(x)&\rightarrow(f*g)(x)=\exp\left(i\theta_{ij}\partial_{x_i}\partial_{x_j}\right)f(x_i)g(x_j)\\ &\approx fg(x)-\frac{i}{2}\theta^{ij}\partial_i^x f\,\partial_j^x g\Big|_{x_i=x_j}+O(\theta^2)\end{aligned} \quad (3)$$

Where, $O(\theta^2)$ stands for second- and higher-order terms of θ. The second in the above equation presents the effects of (space-space) noncommutativity properties. However, the new operators $\hat{\xi}_{iH}(t)=(\hat{x}_i\vee\hat{p}_i)(t)$ and $\hat{\xi}_{iI}(t)=(\hat{x}_{Ii}\vee\hat{p}_{Ii})(t)$ in (HP and IP, respectively) are depending on the corresponding new operator $\hat{\xi}_{iS}=[\hat{x}_i\vee\hat{p}_i]$ in SP from the following projections relations:

$$\begin{cases}\xi_{iH}(t)=\exp\left(i\hat{H}_r^{yp}(t-t_0)\right)\xi_{iS}\exp\left(-i\hat{H}_r^{yp}(t-t_0)\right)\\ \xi_{iI}(t)=\exp\left(i\hat{H}_{or}^{yp}(t-t_0)\right)\xi_{iS}\exp\left(-i\hat{H}_{or}^{yp}(t-t_0)\right)\end{cases}\Rightarrow$$
$$\begin{cases}\hat{\xi}_{iH}(t)=\exp\left(i\hat{H}_{nc-r}^{yp}(t-t_0)\right)*\hat{\xi}_{iS}*\exp\left(-i\hat{H}_{nc-r}^{yp}(t-t_0)\right)\\ \hat{\xi}_{iI}(t)=\exp\left(i\hat{H}_{nc-or}^{yp}(t-t_0)\right)*\hat{\xi}_{iS}*\exp\left(-i\hat{H}_{nc-or}^{yp}(t-t_0)\right)\end{cases} \quad (4)$$

Here, $\xi_{iS}=(x_i\vee p_i)$, $\xi_{iH}(t)=(x_i\vee p_i)(t)$ and $\xi_{iI}(t)=(x_{Ii}\vee p_{Ii})(t)$ are the three representations in relativistic quantum mechanics, while the dynamics of new systems $\frac{\mathrm{d}\xi_{iH}(t)}{\mathrm{dt}}$ are described from the following motion equations in RNCQM:

$$\frac{\mathrm{d}\xi_{iH}(t)}{\mathrm{dt}}=\left[\xi_{iH}(t),\hat{H}_r^{yp}\right]\Rightarrow\frac{\mathrm{d}\hat{\xi}_H(t)}{\mathrm{dt}}=\left[\hat{\xi}_{iH}(t)\overset{*}{,}\hat{H}_{nc-r}^{yp}\right] \quad (5)$$

the operators $\hat{H}_{or}^{yp}$ and $\hat{H}_r^{yp}$ are the unperturbed and global Hamiltonian in RQM for an unequal mixture of scalar and vector Yukawa potentials while $\hat{H}_{nc-or}^{yp}$ and $\hat{H}_{nc-r}^{yp}$ the corresponding Hamiltonians for MUSVY potentials in the RNCQM. The indices are $(i,j\equiv\overline{1,3})$ This paper consists of four sections and the organization scheme is as follows: In the next section, the theory part, we briefly review the Klein-Gordon equation with unequal mixture scalar vector Modified Yukawa potentials based on ref. [17]. Section 3 is devoted to studying the MKGE by applying the Bopp's shift method and the obtained effective potential. Then, we apply the standard perturbation theory to find the energy shift of the ground state, the first excited state and the n^{th} excited state, which is produced by the effects of modified spin-orbital and modified Zeeman interactions. We discuss the nonrelativistic limits. Finally, in the last section, a summary and conclusions are presented.

2. Overview of Energy Eigen-functions and Eigen-values for Unequal Mixture of Scalar and Vector Yukawa Potentials in RQM

As already mentioned our aim is to obtain the spectrum of MKGE with a modified mixture of scalar $S(\hat{r})$ and vector $V(\hat{r})$ Yukawa in (RNC: 3D-RSP) symmetries, we need to revise the corresponding mixture of scalar $S(r)$ and vector $V(r)$ Yukawa in symmetries of ordinary relativistic quantum mechanics [17]:

$$V(r)=-\frac{V_0\exp(-ar)}{r}\text{ and }S(r)=-\frac{S_0\exp(-ar)}{r} \quad (6)$$

Where, $S(r)=\beta V(r)$, $V_0=\alpha Z$ (α and Z are the fine-structure constant and the atomic number), a is the screening parameter and β is an arbitrary constant demonstrating the ratio of the scalar potential to the vector potential. To achieve the goal it is useful to summarize, the KGE with equal scalar and vector potentials for a particle of rest mass M in three-dimensional relativistic quantum mechanics 3D-RQM [17, 50]:

$$\left\{-\nabla^2+(M+S(r))^2-(E-V(r))^2\right\}\Psi(r,\theta,\varphi)=0\Rightarrow\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\left(E_{nl}^{\;2}-M^2\right)-2\left(E_{nl}V(r)+MS(r)\right)+V^2(r)-S^2(r)-\frac{l(l+1)}{r^2}\right\}R_{nl}(r)=0 \quad (7)$$

Where, $\Psi(r,\theta,\varphi)=R_{nl}(r)Y_l^m(\theta,\varphi)$ denotes the complete wave function and ∇^2 is the 3-dimensional Laplacian operator. To eliminate the first order derivative, we introduce a new radial wave function $U_{nl}(r)=rR_{nl}(r)$, thus Eqn. (6) becomes:

$$R_{nl}(r)=\frac{U_{nl}(r)}{r}\Rightarrow\left\{\frac{d^2}{dr^2}-\left(M^2-E_{nl}^{\;2}\right)-2\left(E_{nl}V(r)+MS(r)\right)+V^2(r)-S^2(r)-\frac{l(l+1)}{r^2}\right\}U_{nl}(r)=0 \quad (8)$$

If we introduce the shorthand notation $V_{eff}(r)\equiv 2\left(E_{nl}V(r)+MS(r)\right)-V^2(r)+S^2(r)+\frac{l(l+1)}{r^2}$ and $E_{eff}\equiv M^2-E_{nl}^2$, thus Eqn. (8) reduced to the simple form

$$\left\{\frac{d^2}{dr^2}-\left(E_{eff}+V_{eff}(r)\right)\right\}U_{nl}(r)=0 \quad (9)$$

The reference [17] gives the complete wave function, as a function of the Jacobi polynomial and the spherical harmonic functions:

$$\Psi(r,\theta,\varphi)=N_{nl}\frac{s^{a_{nl}/2}}{r}(1-s)^{(c_l-1)/2}P_n^{(a_{nl},c_l)}(1-2s)Y_l^m(\theta,\varphi) \quad (10)$$

Here $s=\exp(-2ar)$, $c_l=\sqrt{4\left(S_0^2-V_0^2\right)+(2l+1)^2}$, $a_{nl}=\frac{\sqrt{M^2-E_{nl}^2}}{a}$

and N_{nl} is the normalization constant. Therefore, [17] gives the discrete energy eigenvalues of the unequal scalar vector modified Yukawa potentials as a function of the principal quantum number and angular momentum quantum number l

$$\left(2n+1+\sqrt{(2l+1)^2+4\left(S_0^2-V_0^2\right)}+\frac{1}{a}\sqrt{M^2-E_{nl}^2}\right)^2= =-\left(\frac{E_{nl}}{a}-2V_0\right)^2+\left(\frac{M}{a}+2S_0\right)^2 \quad (11)$$

3. Solution of MKGE for MUSVY Potentials

In this section, we shall give a brief overview for the modified mixture unequal scalar vector Yukawa potentials in (RNC: 3D-RS) symmetries. To perform this task, for the physical form of MKGE it is necessary to apply the notion of the Weyl Moyal star product, which we saw previously in the Eqn. (3), on the differential equation that satisfied by the radial wave function $U_l(r)$ in Eqn. (7). Thus, the radial wave function $U_l(r)$ in (RNC: 3D-RS) symmetries become [28-31]

$$\begin{gathered}\left\{\frac{d^2}{dr^2}-\left(M^2-E_{nl}^2\right)-2\left(E_{nl}V(r)+MS(r)\right)+V^2(r)-S^2(r)-\frac{l(l+1)}{r^2}\right\}U_{nl}(r)=0\\ \Rightarrow\left\{\frac{d^2}{dr^2}-\left(M^2-E_{nl}^2\right)-2\left(E_{nl}V(r)+MS(r)\right)+V^2(r)-S^2(r)-\frac{l(l+1)}{r^2}\right\}*U_{nl}(r)=0\end{gathered} \quad (12)$$

The Bopp shift allowed us to simplify the calculation in both NRNCQM and RNCQM, this method has been successfully applied to RNCQM and NRNCQM problems using modified Dirac equation MDE, MKGE and modified Schrödinger equation MSE. This method has produced very promising results for a number of situations having physical and chemical interest. The method reduces three modified fundamental equations (MDE, MKGE and MSE) to the (DE, KGE and SE), respectively, under the simultaneous translation in space. It based on the following new commutator [30-38]:

$$\left[\hat{x}_\mu,\hat{x}_\nu\right]=\left[\hat{x}_\mu(t),\hat{x}_\nu(t)\right]=i\theta_{\mu\nu} \quad (13)$$

The new generalized positions and momentum coordinates $(\hat{x}_\mu,\hat{p}_\nu)$ in the (RNC: 3D-RS) are defined in terms of the commutative counterparts (x_μ,p_ν) in RQM via, respectively [27-35]:

$$(x_\mu,p_\nu)\Rightarrow(\hat{x}_\mu,\hat{p}_\nu)=\left(x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu,p_\mu\right) \quad (14)$$

The above equation allows us to obtain the operator $r^2 \Rightarrow \hat{r}^2 = r^2 - \vec{\mathbf{L}}\vec{\Theta}$ in (RNC-3D: RS) symmetries [51-55]. The two couplings $\vec{\mathbf{L}}\vec{\Theta}$ equal $\left(L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13}\right)$ and (L_x, L_y and L_z) are the three components of angular momentum operator $\vec{L}$ while $\Theta_{\mu\nu} = \theta_{\mu\nu}/2$. Thus, the reduced like Schrödinger equation (without star product) can be written as:

$$\left\{\frac{d^2}{dr^2} - \left(M^2 - E_{nl}^2\right) - 2\left(E_{nl}V(r) + MS(r)\right) + V^2(r) - S^2(r) - \frac{l(l+1)}{r^2}\right\} * U_{nl}(r) = 0 \Rightarrow$$
$$\left\{\frac{d^2}{dr^2} - \left(M^2 - E_{nl}^2\right) - 2\left(E_{nl}V(\hat{r}) + MS(\hat{r})\right) + V^2(\hat{r}) - S^2(\hat{r}) - \frac{l(l+1)}{\hat{r}^2}\right\} U_{nl}(r) = 0 \quad (15)$$

The new operators of $V(\hat{r})$ and $S(\hat{r})$ can be expressed as [28-31]:

$$V(\hat{r}) \equiv V\left(\sqrt{\left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu\right)\left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu\right)}\right) = V(r) - \frac{\vec{\mathbf{L}}\vec{\Theta}}{2r}\frac{\partial V(r)}{\partial r} + O\left(\Theta^2\right)$$
$$S(\hat{r}) \equiv S\left(\sqrt{\left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu\right)\left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu\right)}\right) = S(r) - \frac{\vec{\mathbf{L}}\vec{\Theta}}{2r}\frac{\partial S(r)}{\partial r} + O\left(\Theta^2\right) \quad (16)$$

After straightforward calculations, we can obtain the important terms ($V^2(\hat{r})$ and $S^2(\hat{r})$) which will be used to determine modified unequal scalar vector Modified Yukawa potentials in (RNC: 3D- RSP) symmetries as:

$$V^2(\hat{r}) = V^2(r) - \frac{V(r)}{r}\frac{\partial V(r)}{\partial r}\vec{\mathbf{L}}\vec{\Theta} + O\left(\Theta^2\right)$$
$$S^2(\hat{r}) = S^2(r) - \frac{S(r)}{r}\frac{\partial S(r)}{\partial r}\vec{\mathbf{L}}\vec{\Theta} + O\left(\Theta^2\right) \quad (17)$$

We have

$$\frac{\partial V(r)}{\partial r} = \frac{V_0 a \exp(-ar)}{r} + \frac{V_0 \exp(-ar)}{r^2}, \quad (18.1)$$
$$\frac{\partial S(r)}{\partial r} = \frac{S_0 a \exp(-ar)}{r} + \frac{S_0 \exp(-ar)}{r^2}, \quad (18.2)$$
$$\frac{1}{\hat{r}^2} \approx \frac{1}{\hat{r}^2} + \frac{\vec{\mathbf{L}}\vec{\Theta}}{r^4} + O\left(\Theta^2\right). \quad (18.3)$$

This allows us to write the modified radial part of MKGE in (RNC: 3D-RS) symmetries as:

$$\left\{\frac{d^2}{dr^2} - \left(M^2 - E_{nl}^2\right) - 2\left(E_{nl}V(\hat{r}) + MS(\hat{r})\right) + V^2(\hat{r}) - S^2(\hat{r}) - \frac{l(l+1)}{\hat{r}^2}\right\} U_{nl}(r) = 0 \quad (19)$$

Moreover, to illustrate this equation in a simple mathematical way, it is useful to enter the following symbol $V_{pert}(r)$, thus the radial Eqn. (19) becomes:

$$\left\{\frac{d^2}{dr^2} - \left[E_{eff} + V_{eff}(r) + V_{pert}^{yp}(r)\right]\right\} U_{nl}(r) = 0, \quad (20)$$

With

$$V_{pert}^{yp}(r) = \left[\frac{l(l+1)}{r^4} - \left(\frac{E}{2r}\frac{\partial V(r)}{\partial r} + \frac{E}{2r}\frac{\partial S(r)}{\partial r} + \frac{V(r)}{r}\frac{\partial V(r)}{\partial r} - \frac{S(r)}{r}\frac{\partial S(r)}{\partial r}\right)\right]\vec{\mathbf{L}}\vec{\Theta} \quad (21)$$

Now substituting Eqns. (6) and (18) into Eqn. (19), we find $V_{pert}^{yp}(r)$ in (RNC: 3D-RSP) symmetries as follows:

$$V_{pert}^{yp}(r) = \left\{\begin{array}{l}\frac{l(l+1)}{r^4} - E_{nl}k_0\left(\frac{a\exp(-ar)}{2r^2} + \frac{\exp(-ar)}{2r^3}\right) \\ + \beta_0\left(\frac{a\exp(-2ar)}{r^3} + \frac{\exp(-2ar)}{r^4}\right)\end{array}\right\}\vec{\mathbf{L}}\vec{\Theta} \quad (22)$$

Where, $k_0 \equiv V_0 + S_0$ and $\beta_0 \equiv V_0^2 - S_0^2$. The additive part of the effective potential is proportional to the infinitesimal vector $\vec{\Theta} = \Theta_{11}e_x + \Theta_{12}e_y + \Theta_{13}e_z$ .Thus, we can consider $V_{pert}(r)$ as a perturbation term compared with the parent potential (effective potential operator) $V_{pert}^{yp}(r)$ in (NC: 3D-RS) symmetries. Eqn. (20) can not be solved analytically for any state because of the centrifugal term and the studied potential itself. Therefore, in the present work, we considered the following approximation type suggested by (Greene and Aldrich) and Dong *et al*. [56-58]:

$$\frac{1}{r^2} \approx \frac{4a^2 \exp(-2ar)}{(1-\exp(-2ar))^2} \Rightarrow$$

$$\begin{cases} \dfrac{1}{r^4} \approx \dfrac{16a^4 s^2}{(1-s)^4}, & \dfrac{\exp(-ar)}{r^2} \approx \dfrac{4a^2 s^{3/2}}{(1-s)^2} \\ \dfrac{\exp(-ar)}{r^3} \approx \dfrac{8a^3 s^2}{(1-s)^3}, & \dfrac{\exp(-2ar)}{r^3} \approx \dfrac{8a^3 s^{5/2}}{(1-s)^3} \\ \dfrac{\exp(-2ar)}{r^4} \approx \dfrac{16a^4 s^3}{(1-s)^4} \end{cases} \tag{23}$$

$$\begin{aligned} \langle n|r^{-4}|n\rangle &= 16a^4 N_{nl}^2 \int_0^{+\infty} s^{a_{nl}+2}(1-s)^{c_l-5}\left[P_n^{(a_{nl},c_l)}(1-2s)\right]^2 dr \\ \left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle &= 4a^2 N_{nl}^2 \int_0^{+\infty} s^{a_{nl}+3/2}(1-s)^{c_l-3}\left[P_n^{(a_{nl},c_l)}(1-2s)\right]^2 dr \\ \left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle &= 8a^3 N_{nl}^2 \int_0^{+\infty} s^{a_{nl}+2}(1-s)^{c_l-4}\left[P_n^{(a_{nl},c_l)}(1-2s)\right]^2 dr \\ \left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle &= 8a^3 N_{nl}^2 \int_0^{+\infty} s^{a_{nl}+5/2}(1-s)^{c_l-4}\left[P_n^{(a_{nl},c_l)}(1-2s)\right]^2 dr \\ \left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle &= 16a^4 N_{nl}^2 \int_0^{+\infty} s^{a_{nl}+3}(1-s)^{c_l-5}\left[P_n^{(a_{nl},c_l)}(1-2s)\right]^2 dr \end{aligned} \tag{24}$$

We have used the orthogonality property of the spherical harmonics $\int Y_l^m(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi = \delta_{ll'}\delta_{mm'}$. We have $s = \exp(-2ar)$, this allows us to obtain $dr = -\frac{1}{2a}\frac{ds}{s}$. After introducing a new variable $z = 1-2s$, we have ($dr = \frac{1}{2a}\frac{dz}{1-z}$ and $1-s = \frac{z+1}{2}$). This allows us to reformulating of Eqn. (24) as follows:

$$\begin{aligned} \langle n|r^{-4}|n\rangle &= \frac{16a^3 N_{nl}^2}{2^{a_{nl}+c_l-2}} \int_0^{+\infty} (1-z)^{a_{nl}+1}(1+z)^{c_l-5}\left[P_n^{(a_{nl},c_l)}(z)\right]^2 dz, \\ \left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle &= \frac{4aN_{nl}^2}{2^{a_{nl}+c_l-1/2}} \int_0^{+\infty} (1-z)^{a_{nl}+1/2}(1+z)^{c_l-3}\left[P_n^{(a_{nl},c_l)}(z)\right]^2 dz \\ \left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle &= \frac{8a^2 N_{nl}^2}{2^{a_{nl}+c_l-1}} \int_0^{+\infty} (1-z)^{a_{nl}+1}(1+z)^{c_l-4}\left[P_n^{(a_{nl},c_l)}(z)\right]^2 dz, \\ \left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle &= \frac{8a\ N_{nl}^2}{2^{a_{nl}+c_l-1/2}} \int_0^{+\infty} (1-z)^{a_{nl}+3/2}(1+z)^{c_l-4}\left[P_n^{(a_{nl},c_l)}(z)\right]^2 dz \\ \left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle &= \frac{16a^3 N_{nl}^2}{2^{a_{nl}+c_l-1}} \int_0^{+\infty} (1-z)^{a_{nl}+2}(1+z)^{c_l-5}\left[P_n^{(a_{nl},c_l)}(z)\right]^2 dz \end{aligned} \tag{25}$$

For the ground state ($n = 0$), we have $P_0^{(a_{0l},c_l)}(z) = 1$, thus the above 5-expectation values in Eqn. (25) reduced to the following simple form:

We have the 5-expectation values as:

The purpose here is to give a complete prescription for determing the energy level of the n^{th} excited state, by applying the perturbative theory, in the case of RNCQM. In the first-order perturbation theory the expectation values of $\frac{1}{r^4}$, $\frac{\exp(-ar)}{r^2}$, $\frac{\exp(-ar)}{r^3}$, $\frac{\exp(-2ar)}{r^3}$ and $\frac{\exp(-2ar)}{r^4}$ with respect to the exact solution of Eq. (10), are given by:

$$\begin{aligned} \langle 0|r^{-4}|0\rangle &= \frac{16a^3 N_{0l}^2}{2^{a_{0l}+c_l-2}} \int_0^{+\infty} (1-z)^{a_{0l}+1}(1+z)^{c_l-5} dz \\ \left\langle 0\left|\frac{\exp(-ar)}{r^2}\right|0\right\rangle &= \frac{4aN_{0l}^2}{2^{a_{0l}+c_l-1/2}} \int_0^{+\infty} (1-z)^{a_{0l}+1/2}(1+z)^{c_l-3} dz \\ \left\langle 0\left|\frac{\exp(-ar)}{r^3}\right|0\right\rangle &= \frac{8a^2 N_{0l}^2}{2^{a_{0l}+c_l-1}} \int_0^{+\infty} (1-z)^{a_{0l}+1}(1+z)^{c_l-4} dz \\ \left\langle 0\left|\frac{\exp(-2ar)}{r^3}\right|0\right\rangle &= \frac{8a^2 N_{0l}^2}{2^{a_{0l}+c_l-1/2}} \int_0^{+\infty} (1-z)^{a_{0l}+3/2}(1+z)^{c_l-4} dz \\ \left\langle 0\left|\frac{\exp(-2ar)}{r^4}\right|0\right\rangle &= \frac{16a^3 N_{0l}^2}{2^{a_{0l}+c_l-1}} \int_0^{+\infty} (1-z)^{a_{0l}+2}(1+z)^{c_l-5} dz \end{aligned} \tag{26}$$

Where, $a_{0l} = \frac{\sqrt{M^2 - E_{0l}^2}}{a}$ and E_{0l} is obtained from the following equation:

$$\left(1+\sqrt{(2l+1)^2 + 4(S_0^2 - V_0^2)} + \frac{1}{a}\sqrt{M^2 - E_{0l}^2}\right)^2 = -\left(\frac{E_{0l}}{a} - 2V_0\right)^2 + \left(\frac{M}{a} + 2S_0\right)^2 \tag{27}$$

Comparing Eqn. (26) with the integral of the form [59]:

$$\begin{aligned} &\int_{-1}^{+1} (1-p)^{\alpha}(1+p)^{\beta} P_m^{(\alpha,\beta)} P_n^{(\alpha,\beta)} dp = 2^{\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{mn} \Rightarrow \\ &\int_{-1}^{+1} (1-p)^{n+\alpha}(1+p)^{n+\beta} dp = 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)} \text{ for } (n = 0,1,....) \end{aligned} \tag{28}$$

$$\left\langle 0\left|r^{-4}\right|0\right\rangle = 8a^3N_{0l}^2\,\frac{\Gamma(a_{0l}+2)\Gamma(c_l-4)}{(\lambda_{0l}-3)\Gamma(\lambda_{0l}-3)},\quad \left\langle 0\left|\frac{\exp(-ar)}{r^2}\right|0\right\rangle = 2aN_{0l}^2\,\frac{\Gamma(a_{0l}+3/2)\Gamma(c_l-2)}{(\lambda_{0l}-3/2)\Gamma(\lambda_{0l}-3/2)}$$

$$\left\langle 0\left|\frac{\exp(-ar)}{r^3}\right|0\right\rangle = 4a^2N_{0l}^2\,\frac{\Gamma(a_{0l}+2)\Gamma(c_l-3)}{(\lambda_{0l}-2)\Gamma(\lambda_{0l}-2)},\quad \left\langle 0\left|\frac{\exp(-2ar)}{r^3}\right|0\right\rangle = 4a^2N_{0l}^2\,\frac{\Gamma(a_{0l}+5/2)\Gamma(c_l-3)}{(\lambda_{0l}-3/2)\Gamma(\lambda_{0l}-3/2)} \qquad (29)$$

and $$\left\langle 0\left|\frac{\exp(-2ar)}{r^4}\right|0\right\rangle = 8a^3N_{0l}^2\,\frac{\Gamma(a_{0l}+3)\Gamma(c_l-4)}{(\lambda_{0l}-2)\Gamma(\lambda_{0l}-2)}$$

Where, $\lambda_{0l} = a_{0l} + c_l$. For the first excited state ($n = 1$), we have $P_1^{(a_{1l},c_l)}(z) = \sigma_{1l} + \tau_{1l}(1-z)$, where, $\sigma_{1l} = a_{1l} + 1$, $\tau_{1l} = -\frac{a_{1l}+c_l+2}{2}$ and $a_{1l} = \frac{\sqrt{M^2 - E_{1l}^2}}{a}$ while E_{1l} obtained from the following equation:

$$\left(3 + \sqrt{(2l+1)^2 + 4\left(S_0^2 - V_0^2\right)} + \frac{1}{a}\sqrt{M^2 - E_{1l}^2}\right)^2 = -\left(\frac{E_{1l}}{a} - 2V_0\right)^2 + \left(\frac{M}{a} + 2S_0\right)^2 \qquad (30)$$

Thus, the 5-expectation values in Eqn. (25) reduced to the following simple form:

$$\left\langle 1\left|r^{-4}\right|1\right\rangle \equiv T_{11} + T_{12} + T_{13},$$

$$\left\langle 1\left|\frac{\exp(-ar)}{r^2}\right|1\right\rangle \equiv T_{21} + T_{22} + T_{23},$$

$$\left\langle 1\left|\frac{\exp(-ar)}{r^3}\right|1\right\rangle \equiv T_{31} + T_{32} + T_{33}, \qquad (31)$$

$$\left\langle 1\left|\frac{\exp(-2ar)}{r^3}\right|1\right\rangle \equiv T_{41} + T_{42} + T_{43}$$

and $$\left\langle 1\left|\frac{\exp(-2ar)}{r^4}\right|1\right\rangle \equiv T_{51} + T_{52} + T_{53}.$$

Where, the 15-elements T_{ij} are given by:

$$T_{11} = \frac{16a^3N_{1l}^2\sigma_{1l}^2}{2^{a_{1l}+c_l-2}}\int_0^{+\infty}(1-z)^{a_{1l}+1}(1+z)^{c_l-5}\,dz$$

$$T_{12} = \frac{16a^3N_{1l}^2\tau_{1l}^2}{2^{a_{1l}+c_l-2}}\int_0^{+\infty}(1-z)^{a_{1l}+3}(1+z)^{c_l-5}\,dz \quad ,$$

$$T_{13} = \frac{32a^3N_{1l}^2\sigma_{1l}\tau_{1l}}{2^{a_{1l}+c_l-2}}\int_0^{+\infty}(1-z)^{a_{1l}+2}(1+z)^{c_l-5}\,dz$$

$$T_{21} = \frac{4aN_{1l}^2\sigma_{1l}^2}{2^{a_{1l}+c_l-1/2}}\int_0^{+\infty}(1-z)^{a_{1l}+1/2}(1+z)^{c_l-3}\,dz$$

$$T_{22} = \frac{4aN_{1l}^2\tau_{1l}^2}{2^{a_{1l}+c_l-1/2}}\int_0^{+\infty}(1-z)^{a_{1l}+5/2}(1+z)^{c_l-3}\,dz \quad ,$$

$$T_{23} = \frac{8aN_{1l}^2\sigma_{1l}\tau_{1l}}{2^{a_{1l}+c_l-1/2}}\int_0^{+\infty}(1-z)^{a_{1l}+3/2}(1+z)^{c_l-3}\,dz$$

$$T_{31} = \frac{8a^2N_{1l}^2\sigma_{1l}^2}{2^{a_{1l}+c_l-1}}\int_0^{+\infty}(1-z)^{a_{1l}+1}(1+z)^{c_l-4}\,dz$$

$$T_{32} = \frac{8a^2N_{1l}^2\tau_{1l}^2}{2^{a_{1l}+c_l-1}}\int_0^{+\infty}(1-z)^{a_{1l}+3}(1+z)^{c_l-4}\,dz \quad ,$$

$$T_{33} = \frac{16a^2N_{1l}^2\sigma_{1l}\tau_{1l}}{2^{a_{1l}+c_l-1}}\int_0^{+\infty}(1-z)^{a_{1l}+2}(1+z)^{c_l-4}\,dz$$

$$T_{41} = \frac{8a^2N_{1l}^2\sigma_{1l}^2}{2^{a_{1l}+c_l-1/2}}\int_0^{+\infty}(1-z)^{a_{1l}+3/2}(1+z)^{c_l-4}\,dz$$

$$T_{42} = \frac{8a^2N_{1l}^2\tau_{1l}^2}{2^{a_{nl}+c_l-1/2}}\int_0^{+\infty}(1-z)^{a_{nl}+7/2}(1+z)^{c_l-4}\,dz$$

$$T_{43} = \frac{16a^2N_{1l}^2\sigma_{1l}\tau_{1l}}{2^{a_{1l}+c_l-1/2}}\int_0^{+\infty}(1-z)^{a_{1l}+5/2}(1+z)^{c_l-4}\,dz$$

and
$$T_{51} = \frac{16a^3N_{1l}^2\sigma_{1l}^2}{2^{a_{1l}+c_l-1}}\int_0^{+\infty}(1-z)^{a_{1l}+2}(1+z)^{c_l-5}\,dz,\; T_{52} = \frac{16a^3N_{1l}^2\tau_{1l}^2}{2^{a_{1l}+c_l-1}}\int_0^{+\infty}(1-z)^{a_{1l}+4}(1+z)^{c_l-5}\,dz \qquad (32)$$

$$T_{53} = \frac{32a^3N_{1l}^2\sigma_{1l}\tau_{1l}}{2^{a_{1l}+c_l-1}}\int_0^{+\infty}(1-z)^{a_{1l}+3}(1+z)^{c_l-5}\,dz$$

We apply the integral in Eqn. (27) to obtain the 15-elements T_{ij} as follows:

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \\ T_{41} & T_{42} & T_{43} \\ T_{51} & T_{52} & T_{53} \end{pmatrix} = N_{1l}^2 \begin{pmatrix} 8a^3\sigma_{1l}^2 \frac{\Gamma(a_{1l}+2)\Gamma(c_l-4)}{(\delta_{1l}-3)\Gamma(\delta_{1l}-3)} & 32a^3\tau_{1l}^2 \frac{\Gamma(a_{1l}+4)\Gamma(c_l-4)}{(\delta_{1l}-1)\Gamma(\delta_{1l}-1)} & 32a^3\sigma_{1l}\tau_{1l} \frac{\Gamma(a_{1l}+3)\Gamma(c_l-4)}{(\delta_{1l}-2)\Gamma(\delta_{1l}-2)} \\ 2a\sigma_{1l}^2 \frac{\Gamma(a_{1l}+3/2)\Gamma(c_l-2)}{(\delta_{1l}-3/2)\Gamma(\delta_{1l}-3/2)} & 8a\tau_{1l}^2 \frac{\Gamma(a_{1l}+7/2)\Gamma(c_l-2)}{(\delta_{1l}+1/2)\Gamma(\delta_{1l}+1/2)} & 8a\sigma_{1l}\tau_{1l} \frac{\Gamma(a_{1l}+5/2)\Gamma(c_l-2)}{(\delta_{1l}-1/2)\Gamma(\delta_{1l}-1/2)} \\ 4a^2\sigma_{1l}^2 \frac{\Gamma(a_{1l}+2)\Gamma(c_l-3)}{(\delta_{1l}-2)\Gamma(\delta_{1l}-2)} & 16a^2\tau_{1l}^2 \frac{\Gamma(a_{1l}+4)\Gamma(c_l-3)}{\delta_{1l}\Gamma(\delta_{1l})} & 16a^2\sigma_{1l}\tau_{1l} \frac{\Gamma(a_{1l}+3)\Gamma(c_l-3)}{(\delta_{1l}-1)\Gamma(\delta_{1l}-1)} \\ 4a^2\sigma_{1l}^2 \frac{\Gamma(a_{1l}+5/2)\Gamma(c_l-3)}{(\delta_{1l}-3/2)\Gamma(a_{1l}+c_l-3/2)} & 16a^2\tau_{1l}^2 \frac{\Gamma(a_{1l}+9/2)\Gamma(c_l-3)}{(\delta_{1l}+1/2)\Gamma(\delta_{1l}+1/2)} & 16a^2N\sigma_{1l}\tau_{1l} \frac{\Gamma(a_{1l}+7/2)\Gamma(c_l-3)}{(\delta_{1l}-1/2)\Gamma(\delta_{1l}-1/2)} \\ 8a^3\sigma_{1l}^2 \frac{\Gamma(a_{1l}+3)\Gamma(c_l-4)}{(\delta_{1l}-2)\Gamma(\delta_{1l}-2)} & 16a^3\tau_{1l}^2 \frac{\Gamma(a_{1l}+5)\Gamma(c_l-4)}{\delta_{1l}\Gamma(\delta_{1l})} & 32a^3\sigma_{1l}\tau_{1l} \frac{\Gamma(a_{1l}+4)\Gamma(c_l-4)}{(\delta_{1l}-1)\Gamma(\delta_{1l}-1)} \end{pmatrix} \quad (33)$$

Where, $\delta_{1l} = a_{1l} + c_l$. The present work is two-fold, the first is to correspond to replace $\vec{\mathbf{L}}\vec{\Theta}$ by $\Theta\vec{L}\vec{S}$ ($\Theta = \sqrt{\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2}$), we have chosen the vector $\vec{\Theta}$ parallel to the spin $\vec{S}$ and we replace $\Theta\vec{L}\vec{S}$ by $\frac{\Theta}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$. It is well known, that, the group of operators ($H_{so}^{yp}(r,\theta)$, J^2, L^2, S^2 and J_z) forms a complete set of conserved physics quantities, the eigenvalues of the spin-orbital coupling operator are $k(l) \equiv \frac{1}{2}\{j(j+1) - l(l+1) - s(s+1\}$, with $|l-s| \le j \le |l+s|$. This allows us to obtain the energy shift $\Delta E(n,j,l,s)$ due to the spin-orbital complying induced by $V_{pert}^{yp}(r)$ for the ground state, first excited state in (RNC: 3D-RS) symmetries as follows:

$$\begin{aligned} \Delta E(n=0,j,l,s) &= k(l)\left\{\langle 0|r^{-4}|0\rangle l(l+1) - E_{0l}k_0\left(\frac{a}{2}\left\langle 0\left|\frac{\exp(-ar)}{r^2}\right|0\right\rangle + \frac{1}{2}\left\langle 0\left|\frac{\exp(-ar)}{r^3}\right|0\right\rangle\right) + \beta_0\left(\frac{a}{2}\left\langle 0\left|\frac{\exp(-2ar)}{r^3}\right|0\right\rangle + \left\langle 0\left|\frac{\exp(-2ar)}{r^4}\right|0\right\rangle\right)\right\} \\ \Delta E(n=1,j,l,s) &= k(l)\left\{\langle 1|r^{-4}|1\rangle l(l+1) - E_{1l}k_0\left(\frac{a}{2}\left\langle 1\left|\frac{\exp(-ar)}{r^2}\right|1\right\rangle + \frac{1}{2}\left\langle 1\left|\frac{\exp(-ar)}{r^3}\right|1\right\rangle\right) + \beta_0\left(\frac{a}{2}\left\langle 1\left|\frac{\exp(-2ar)}{r^3}\right|1\right\rangle + \left\langle 1\left|\frac{\exp(-2ar)}{r^4}\right|1\right\rangle\right)\right\} \end{aligned} \quad (34)$$

Which can be generalized easily to the n^{th} excited states in (RNC: 3D-RS) symmetries as follows:

$$\Delta E(n,j,l,s) = k(l)\left\{\langle n|r^{-4}|n\rangle l(l+1) - E_{nl}k_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle + \frac{1}{2}\left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle\right) + \beta_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle + \left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle\right)\right\} \quad (35)$$

The second is corresponding to replace both ($\vec{\mathbf{L}}\vec{\Theta}$ and Θ_{12}) by ($\sigma_{12}BL_z$ and $\sigma_{12}B$) in addition to use this relation $\langle n,l,m|L_z|n',l',m'\rangle = m'\delta_{nn'}\delta_{ll'}\delta_{mm'}$ (Here, $-l' \le m' \le +l'$). This allows us to obtain the energy shift $\Delta E(n,m)$ due to the modified Zeeman effect induced by $V_{pert}^{yp}(r)$ for the ground state, the first excited state in (RNC: 3D-RS) symmetries as follows:

$$\begin{aligned} \Delta E(n=0,m) &= B\left\{\langle 0|r^{-4}|0\rangle l(l+1) - E_{0l}k_0\left(\frac{a}{2}\left\langle 0\left|\frac{\exp(-ar)}{r^2}\right|0\right\rangle + \frac{1}{2}\left\langle 0\left|\frac{\exp(-ar)}{r^3}\right|0\right\rangle\right) + \beta_0\left(\frac{a}{2}\left\langle 0\left|\frac{\exp(-2ar)}{r^3}\right|0\right\rangle + \left\langle 0\left|\frac{\exp(-2ar)}{r^4}\right|0\right\rangle\right)\right\}\sigma m \\ \Delta E(n=1,m) &= B\left\{\langle 1|r^{-4}|1\rangle l(l+1) - E_{1l}k_0\left(\frac{a}{2}\left\langle 1\left|\frac{\exp(-ar)}{r^2}\right|1\right\rangle + \frac{1}{2}\left\langle 1\left|\frac{\exp(-ar)}{r^3}\right|1\right\rangle\right) + \beta_0\left(\frac{a}{2}\left\langle 1\left|\frac{\exp(-2ar)}{r^3}\right|1\right\rangle + \left\langle 1\left|\frac{\exp(-2ar)}{r^4}\right|1\right\rangle\right)\right\}\sigma m \end{aligned} \quad (36)$$

Which can be generalized easily to the n^{th} excited states in (NC: 3D-RS) symmetries as follows:

$$\Delta E(n,m) = B\left\{\langle n|r^{-4}|n\rangle l(l+1) - E_{nl}k_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle + \frac{1}{2}\left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle\right) + \beta_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle + \left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle\right)\right\}\sigma m \quad (37)$$

The superposition principle permitted to deduce the additive energy shift $\Delta E(n,j,l,s,m)$ due to the spin-orbital complying and modified Zeeman effect which induced by $V_{pert}^{yp}(r)$ for the ground state, the first excited state in (RNC: 3D-RS) symmetries as follows:

$$\Delta E(n=0,j,l,s,m)=\left\{\langle 0|r^{-4}|0\rangle l(l+1)-E_{0l}k_0\left(\frac{a}{2}\left\langle 0\left|\frac{\exp(-ar)}{r^2}\right|0\right\rangle+\frac{1}{2}\left\langle 0\left|\frac{\exp(-ar)}{r^3}\right|0\right\rangle\right)+\beta_0\left(\frac{a}{2}\left\langle 0\left|\frac{\exp(-2ar)}{r^3}\right|0\right\rangle+\left\langle 0\left|\frac{\exp(-2ar)}{r^4}\right|0\right\rangle\right)\right\}\{k(l)\Theta+B\sigma m\} \quad (38)$$

$$\Delta E(n=1,j,l,s,m)=\left\{\langle 1|r^{-4}|1\rangle l(l+1)-E_{0l}k_0\left(\frac{a}{2}\left\langle 1\left|\frac{\exp(-ar)}{r^2}\right|1\right\rangle+\frac{1}{2}\left\langle 1\left|\frac{\exp(-ar)}{r^3}\right|0\right\rangle\right)+\beta_0\left(\frac{a}{2}\left\langle 1\left|\frac{\exp(-2ar)}{r^3}\right|1\right\rangle+\left\langle 1\left|\frac{\exp(-2ar)}{r^4}\right|1\right\rangle\right)\right\}\{k(l)\Theta+B\sigma m\}$$

Which can be generalized easily to the n^{th} excited states in (RNC: 3D-RS) symmetries as follows:

$$\Delta E(n,j,l,s,m)=\left\{\langle n|r^{-4}|n\rangle l(l+1)-E_{0l}k_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle+\frac{1}{2}\left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle\right)+\beta_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle+\left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle\right)\right\}\{k(l)\Theta+B\sigma m\} \quad (39)$$

When we look to the expressions of effective MUSVY potentials $V_{pert}^{yp}(r)$ and effective energy $\Delta E(n,j,l,s,m)$, it is clear that have a carry unit of energy, thus we can deduces explicitly the energy $E_{r-nc}^{y}(a,S_0,V_0,n=0,j,l,m)$, $E_{r-nc}^{y}(a,S_0,V_0,n=1,j,l,m)$ and $E_{r-nc}^{y}(a,S_0,V_0,n,j,l,m)$ corresponding the ground state, the first excited state, and the n^{th} excited state, respectively, as a functions of the shift energy $\Delta E(n,j,l,s,m)$ and (E_{0l}, E_{1l} and E_{nl}) in (RNC: 3D-RS) symmetries as follows:

$$E_{r-nc}^{y}(a,S_0,V_0,n=0,j,l,m)=E_{0l}+[\Delta E(n=0,j,l,s,m)]^{1/2} \quad (40.1)$$

$$\begin{aligned}E_{r-nc}^{y}(a,S_0,V_0,n=1,j,l,m)&=E_{1l}+[\Delta E(n=1,j,l,s,m)]^{1/2}\\E_{r-nc}^{y}(a,S_0,V_0,n,j,l,m)&=E_{nl}+[\Delta E(n,j,l,s,m)]^{1/2}\end{aligned} \quad (40.2)$$

Where E_{0l}, E_{1l} and E_{nl} are obtained from Eqns. (27), (30) and (11), respectively. Now, it is important to apply our results obtained to the case of a bosonic particle with spin one ($\vec{S}=\vec{1}$), we have $|l-1|\leq j\leq |l+1|$, allows us to obtain three values of j ($j=l\pm 1,l$) that gives $(k_1(l),k_2(l),k_3(l))\equiv\frac{1}{2}(l,-2,-2l-2)$ and thus we obtain three values of the energy shift as follows:

$$\begin{aligned}\Delta E(n,j=l+1,l,s,m)&=\left\{\langle n|r^{-4}|n\rangle l(l+1)-E_{0l}k_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle+\frac{1}{2}\left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle\right)+\beta_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle+\left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle\right)\right\}\left\{\frac{l}{2}\Theta+B\sigma m\right\}\\\Delta E(n,j=l,l,s,m)&=\left\{\langle n|r^{-4}|n\rangle l(l+1)-E_{0l}k_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle+\frac{1}{2}\left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle\right)+\beta_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle+\left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle\right)\right\}\{-\Theta+B\sigma m\}\\\Delta E(n,j=l-1,l,s,m)&=\left\{\langle n|r^{-4}|n\rangle l(l+1)-E_{0l}k_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle+\frac{1}{2}\left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle\right)+\beta_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle+\left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle\right)\right\}\{-(l+1)\Theta+B\sigma m\}\end{aligned} \quad (41)$$

The above results of the degenerate energy shift and Eqn. (40) gives the energy of a bosonic particle with $\vec{S}=\vec{1}$ under the MUSVY potentials as follows:

$$E_{r-nc}^{y}(a,S_0,V_0,n,j,l,m)=E_{nl}+\begin{cases}\pm[\Delta E(n,j=l+1,l,s=1,m)]^{1/2} & \text{for} \quad j=l+1\\\pm[\Delta E(n,j,l,s=1,m)]^{1/2} & \text{for} \quad j=l\\\pm E[\Delta E(n,j=l-1,l,s=1,m)]^{1/2} & \text{for} \quad j=l-1\end{cases} \quad (42)$$

On the other hand, it is evident to consider the quantum number m takes $(2l+1)$ values and we have also two values for $j=l\pm 1,l$, thus every state in usually three-dimensional space of energy for bosonic particles with $\vec{S}=\vec{1}$ under MCIQP will become double $3(2l+1)$ sub-states. To obtain the total complete degeneracy of energy level of the MUSVY potentials in (NC-3D: RSP) symmetries, we need to sum for all allowed values of l. Total degeneracy is thus,

$$\underbrace{2\sum_{i=0}^{n-1}(2l+1)=2n^2}_{\text{In}-R\text{QM}}\rightarrow\underbrace{3\left(2\sum_{i=0}^{n-1}(2l+1)\right)\equiv 6n^2}_{\text{In}-R\text{NCQM}} \quad (43)$$

It's clear that the degeneracy of the initial spectral line is broken and it was replaced by a more precise and one. The doubled of the total complete degeneracy of energy level for bosonic particles with $\vec{S}=\vec{1}$, in RNCQM symmetries under the MUSVY potentials gives a clear physical indicator by showing that the physical treatment with RNCQM appear more detailed and clear if compared with similar energy levels obtained in ordinary relativistic quantum mechanics. In order to consider further the interpretation of the positive and negative energy solutions of the modified Klein-Gordon equation one can consider the nonrelativistic limit. For this purpose, we apply the following transformations:

$$\begin{aligned} &E^{y}_{r-nc}(a,S_0,V_0,n,j,l,m)-M \rightarrow E^{y}_{nr-nc}(a,S_0,V_0,n,j,l,m) \\ &2E^{y}_{r-nc}(a,S_0,V_0,n,j,l,m)+M \rightarrow 2\mu \end{aligned} \tag{44}$$

Here μ is the reduced mass of the electron e and the atom Ze and $E^{y}_{nr-nc}(a,S_0,V_0,n,j,l,m)$ is the non-relativistic energy, inserting above transformation into Eqn. (40) yields:

$$\begin{aligned} E^{y}_{nr-nc}(a,S_0,V_0,n,j,l,m) &= -\frac{a^2}{2\mu}\left(\frac{\mu V_0}{a(n+l+1)}-(n+l+1)\right)^2 \\ &-2\mu+\left[\Delta E_{nr}(n,j,l,s,m)\right]^{1/2} \end{aligned} \tag{45}$$

Where the first term in Eqn. (45) is the nonrelativistic energy determined by [3]. In the non-relativistic study, Eqn. (42) applies to hydrogen like atoms such as He^{+}, Be^{+} and Li^{2+}, we have $|l-1/2| \leq j \leq |l+1/2|$, allows us to obtain two values of j ($j=l\pm 1/2$) that gives $(k_1(l),k_2(l)) \equiv \frac{1}{2}(l,-l-1)$ and thus, we obtain two values of the energy shift $\Delta E_{nr}(n,j,l,s,m)$ as follows:

$$\begin{aligned} \Delta E(n,j=l+1/2,l,s,m) &= \left\{\langle n|r^{-4}|n\rangle l(l+1)-E_{0l}k_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle+\frac{1}{2}\left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle\right)+\beta_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle+\left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle\right)\right\}\left\{\frac{l}{2}\Theta+B\sigma m\right\} \\ \Delta E(n,j=l-1/2,l,s,m) &= \left\{\langle n|r^{-4}|n\rangle l(l+1)-E_{0l}k_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-ar)}{r^2}\right|n\right\rangle+\frac{1}{2}\left\langle n\left|\frac{\exp(-ar)}{r^3}\right|n\right\rangle\right)+\beta_0\left(\frac{a}{2}\left\langle n\left|\frac{\exp(-2ar)}{r^3}\right|n\right\rangle+\left\langle n\left|\frac{\exp(-2ar)}{r^4}\right|n\right\rangle\right)\right\}\left\{-\frac{l+1}{2}\Theta+B\sigma m\right\} \end{aligned} \tag{46}$$

The above results of the degenerate energy shift and Eqn. (43) gives the nonrelativistic energy $E^{y}_{nr-nc}(a,S_0,V_0,n,j=l\pm 1/2,l,s=1/2,m)$ of a fermionic particle with $\vec{S}=\overrightarrow{1/2}$ under the MUSVY potentials as follows:

$$E^{y}_{nr-nc}(a,S_0,V_0,n,j,l,m) = -\frac{a^2}{2\mu}\left(\frac{\mu V_0}{a(n+l+1)}-(n+l+1)\right)^2-2\mu+\begin{cases}\left[\Delta E(n,j=l+1/2,l,s=1/2,m)\right]^{1/2} & \text{for } j=l+1/2 \\ \left[\Delta E(n,j=l-1,l,s=1/2,m)\right]^{1/2} & \text{for } j=l-1/2\end{cases} \tag{47}$$

Let us now look the special cases , when $a=0$, $V_0=-Ze^2$ and $S_0=0$, which give the effective Colombian potential in noncommutative space $V^{col}_{pert}(r,a=0,V_0=-Ze^2,S_0=0)$ and the corresponding like radial Schrödinger equation which is compatible with the results of reference [27]:

$$\begin{aligned} &V^{col}_{pert}(r,a=0,V_0=-Ze^2,S_0=0)=\left[\frac{l(l+1)}{r^4}+(E+M)\frac{Ze^2}{r^3}\right]\vec{\mathbf{L}}\vec{\Theta} \\ &\left\{\frac{d^2}{dr^2}+(E^2_{nl}-M^2)-2(E_{nl}+M)\left(-\frac{Ze^2}{r}\right)-\frac{l(l+1)}{r^2}-\vec{\mathbf{L}}\vec{\Theta}\left[\frac{l(l+1)}{r^4}+(E+M)\frac{Ze^2}{r^3}\right]\right\}U_l(r)=0 \end{aligned} \tag{48}$$

Regarding obtained results in Eqs. (38) and (39), the energy shift are depended on the spin non zero, one can deduce that the modified Klein-Gordon which is treated in our paper under MUSVY potentials can be prolonged to describe not only spin-zero particles, but particles with spin 1 for example. Thus one can conclude that the MKG become similar to the Duffin–Kemmer equation. This, however, points towards the nonrelativistic energy spectrum for the unequal mixture scalar and vector Yukawa potentials in the in (RNC: 3D-RS) symmetries. If we consider $(\Theta,\sigma)\rightarrow(0,0)$, we recover the results of commutative space of ref. [17] obtained for the MUSVY potentials, which means that our calculations are correct.

4. Conclusion

In this paper, we have investigated the MKGE for the MUSVY potentials in the non-commutative 3-dimensional spaces. The energy $E^{y}_{r-nc}(a,S_0,V_0,n=0,j,l,m)$,

$E^{y}_{r-nc}(a, S_0, V_0, n=1, j, l, m)$ and $E^{y}_{r-nc}(a, S_0, V_0, n, j, l, m)$ corresponding to the ground state, the first excited state and n^{th} excited state as a function of the shift energy $\Delta E(n, j, l, s, m)$ and E_n due to the noncommutativity property is obtained via first-order perturbation theory and expressed by the Gamma function, the discreet atomic quantum numbers (j, l, s, m) and the potential parameters (a, V_0 and S_0), in addition to non-commutativity parameters (Θ and σ). This behavior is similar to the Zeeman effect and spin-orbit coupling in which a magnetic field is applied to the system and the spin-orbital couplings which are induced with the effect $V^{yp}_{pert}(r)$ in (RNC: 3D-RS) symmetries. Therefore, we can conclude that the MKGE becomes similar to the Duffin–Kemmer equation under MUSVY potentials. It describes the dynamical state of a particle with spin one in the symmetries of RNCQM. We have seen that the physical treatment of MKGE under the MUSVY potentials for bosonic particles with $\vec{S} = \vec{1}$ gives a very clear physical indication by showing that physical treatments with RNCQM appear more detailed and clear if compared with similar energy levels obtained in ordinary relativistic quantum mechanics.

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