Adaptive Synchronization of Identical and Non-Identical Hyperchaotic Systems with Unknown Parameters

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In this paper, adaptive control theory is utilized to derive nonlinear controllers for the synchronization of two identical and non-identical hyperchaotic systems with unknown parameters. Based on the Lyapunov stability theory, the adaptive control laws for synchronization controllers associated with adaptive update laws of system parameters are developed to make the states of two identical and non-identical systems asymptotically synchronized. The feasibility of the obtained results are validated with numerical simulation.

1. Introduction

The study of nonlinear dynamical systems and chaos have become a subject of great interest and it has attracted enormous research interest after the first numerical demonstration of chaotic phenomenon by Lorenz [1]. Chaotic systems are associated with complex dynamical behaviors that possess some special features including bounded trajectories with positive Lyapunov exponents; and sensitive dependence on initial conditions. The study of chaotic dynamics has found applications in various branches of scientific and engineering disciplines, including information processing, power converters, biological systems and chemical reactions etc. Another important area that have been widely studied in nonlinear dynamics is hyperchaotic systems. These are systems that are characterized with more than one positive Lyapunov exponents and thus generates more complex dynamics than the low dimensional chaotic systems. Therefore, they possess broader applications particularly in secure communications wherein the presence of more than one positive Lyapunov exponent have been utilized to improve the security of communications [2].

The year 1990 marked a turning point in the study of chaotic dynamics when Pecora and Carroll [3] put forward the idea that two chaotic systems evolving from different initial conditions can become synchronized. The main idea of synchronization is to make the states of slave system track the states of the master system as the time t approaches infinity. In view of its potential applications in secure communications and surveillance (see [4] and the references therein), the phenomena of synchronization have been extensively studied in the context of laser dynamics, electronics circuits, biological and chemical systems [5]. Till now, different types of synchronization have been observed in the interacting chaotic systems, such as complete synchronization, CS [3], generalized synchronization GS [6], phase synchronization PS [7], lag synchronization LS [8], anti-synchronization AS [9], and so on. For an excellent review of various types of synchronization, the reader is referred to the book by Pikovsky [10]. At the same time, many advanced theories and methodologies have been proposed for controlling chaotic synchronization of some types of chaotic/hyperchaotic attractors. Notably among these approaches are backstepping design methods [11, 12], active control methods [13], adaptive synchronization methods [14–16], linear state error feedback control approach [17], sliding mode control methods [18], and so on. The adaptive control methods for synchronization of chaotic/hyperchaotic systems is based on the numerous variants of adaptive control for chaotic/hyperchaotic systems. The approach has been applied extensively for synchronization of hyperchaotic systems (see Refs. [14, 15] and references therein). In fact, adaptive methods for chaos synchronization have demonstrated some capabilities of extracting information from unknown systems.

The need to evaluate unknown parameters in models of nonlinear physical, biological and engineering systems is of vital importance and relevant for various scientific and engineering applications particularly in model predictions. In order

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to estimate unknown parameters of nonlinear systems, synchronization shows promise in estimating the desired parameters. Furthermore, it has been shown that global identifiability considers the issue of uniquely estimating all the free parameters of a model from experimental data [19]. If a model is non-identifiable, the estimated parameters will lead, irrespective of the applied method, to artifacts in the model calibration and errors in subsequent model predictions. Thus, there is a fundamental need for reliable methods for estimating the unknown parameters in nonlinear systems. In view of the above mentioned points, the present study is therefore geared towards utilizing the adaptive techniques to design some novel nonlinear controllers for chaos synchronization of identical and non-identical hyperchaotic systems with unknown parameters using L*ü* [20] and Lorenz-Stenflo (LS) [14] hyperchaotic systems as paradigms. Based on the Lyapunov theorem of stability and adaptive control laws, nonlinear synchronization controllers and controllers associated with the adaptive update laws of parameters are developed for estimating the unknown parameters in identical and non-identical hyperchaotic systems of interest.

The rest of the paper is structured as follows. In Section 2, adaptive synchronization of identical hyperchaotic L*ü* system with unknown parameters is presented while Section 3 is devoted to adaptive synchronization of hyperchaotic LS systems with unknown parameters. In Section 4, we derive the control laws and numerically demonstrate adaptive synchronization of non-identical L*ü* and LS systems wherein the parameters of the drive L*ü* system are known while the parameters of the response LS system are unknown. Finally, the concluding remark is given in Section 5.

2. Adaptive Synchronization of Identical Hyperchaotic L*ü* System

In this section, we shall present both theoretical and numerical simulation result for adaptive synchronization of two hyperchaotic L*ü* systems evolving from different initial conditions.

2.1. Controller design

Here, some adaptive control laws and parameter update laws for synchronization of identical L*ü* systems [20], where the parameters of both the drive and response systems are unknown.

Let us consider drive hyperchaotic L*ü* system given by [20]:

$$\begin{aligned}
\dot{x}_1 &= -\alpha(x_1 - x_2) + x_4, \\
\dot{x}_2 &= \gamma x_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - \beta x_3, \\
\dot{x}_4 &= x_1 x_3 + \eta x_4
\end{aligned} \tag{1}$$

Where, x_i $(i = 1, 2, 3, 4)$ are the state variables and α, β, γ and η are the unknown parameters of the system.

As the response system, we consider the controlled hyperchaotic L*ü* dynamics described by

$$\begin{aligned}
\dot{y}_1 &= -\alpha(y_1 - y_2) + y_4 + u_1, \\
\dot{y}_2 &= \gamma y_2 - y_1 y_3 + u_2, \\
\dot{y}_3 &= y_1 y_2 - \beta y_3 + u_3, \\
\dot{y}_4 &= y_1 y_3 + \eta y_4 + u_4
\end{aligned} \tag{2}$$

Where, y_i (i=1,2,3,4) are the state variables and u_i are the nonlinear controllers to be determined.

The L*ü* hyperchaotic system governed by Eqn. (1) has been shown to exhibit rich varieties of dynamical behaviour including hyperchaotic motion - depicted in Fig. 1 - with the following parameter settings $\alpha = 36$, $\beta = 3$, $\gamma = 20$ and $\eta = 1.0$

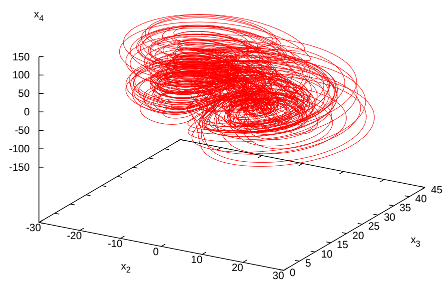

FIG. 1: Phase portrait of the chaotic attractor of the hyperchaotic L*ü* system with the following parameters $\alpha = 36.0$, $\beta = 3.0$, $\gamma = 20.0$ and $\eta = 1.0$.

Let us define the synchronization error as follows

$$e = y_i - x_i. \tag{3}$$

By subtracting Eqn. (1) from Eqn. (2) and applying the definition of error system in Eqn. (3), one readily obtains a time varying error system:

$$\begin{aligned}
\dot{e}_1 &= \alpha(e_2 - e_1) + e_4 + u_1, \\
\dot{e}_2 &= \gamma e_2 - y_1 e_3 - x_3 e_1 + u2, \\
\dot{e}_3 &= y_1 e_2 + x_2 e_1 - \beta e_3 + u3, \\
\dot{e}_4 &= y_1 e_3 + x_3 e_1 + \eta e_4 + u4
\end{aligned} \tag{4}$$

Let us define the adaptive control functions u_i, $(i = 1, 2, 3, 4)$ as follows:

$$\begin{aligned} u_1 &= \alpha_1(e_1 - e_2) - e_4 - k_1 e_1, \\ u_2 &= -\gamma_1 e_2 + y_1 e_3 + x_3 e_1 - k_2 e_2, \\ u_3 &= \beta_1 e_3 - y_1 e_2 - x_2 e_1 - k_3 e_3, \\ u_4 &= -\eta_1 e_4 - y_1 e_3 - x_3 e_1 - k_4 e_4 \end{aligned} \quad (5)$$

Where, α_1, β_1, γ_1 and η_1 are estimates of α, β, γ and η respectively, and k_i $(i = 1, 2, 3, 4)$ are non-negative constants.

By substituting Eqn. (5) into Eqn. (4), we obtain

$$\begin{aligned} \dot{e}_1 &= (\alpha - \alpha_1)(e_2 - e_1) - k_1 e_1, \\ \dot{e}_2 &= (\gamma - \gamma_1)e_2 - k_2 e_2, \\ \dot{e}_3 &= -(\beta - \beta_1)e_3 - k_3 e_3, \\ \dot{e}_4 &= (\eta - \eta_1)e_4 - k_4 e_4 \end{aligned} \quad (6)$$

We define the parameter errors as:

$$e_\alpha = \alpha - \alpha_1, \;\; e_\beta = \beta - \beta_1, \;\; e_\gamma = \gamma - \gamma_1, \;\; e_\eta = \eta - \eta_1 \quad (7)$$

Substituting Eqn. (7) into Eqn. (6), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= e_\alpha(e_2 - e_1) - k_1 e_1, \\ \dot{e}_2 &= e_\gamma e_2 - k_2 e_2, \\ \dot{e}_3 &= -e_\beta e_3 - k_3 e_3, \\ \dot{e}_4 &= e_\eta e_4 - k_4 e_4 \end{aligned} \quad (8)$$

From Eqn. (7) we note that

$$\dot{e}_\alpha = -\dot{\alpha}_1, \;\; \dot{e}_\beta = -\dot{\beta}_1, \;\; \dot{e}_\gamma = -\dot{\gamma}_1, \;\; \dot{e}_\eta = -\dot{\eta}_1 \quad (9)$$

Let us consider a positive definite quadratic Lyapunov function V defined by

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\eta^2) \quad (10)$$

Differentiating Eqn. (10) along the directions of the error trajectories (8) and using Eqn. (9), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + a_1 + a_2 + a_3 + a_4 \quad (11)$$

Where, $a_1 = e_\alpha[e_1(e_2 - e_1) - \dot{\alpha}_1]$, $a_2 = e_\gamma[e_2 e_2 - \dot{\gamma}_1]$, $a_3 = e_\beta[-e_3 e_3 - \dot{\beta}_1]$, $a_4 = e_\eta[-e_4 e_4 - \dot{\eta}_1]$.
Based on Eqn. (11), the estimated parameters are updated by the following laws:

$$\begin{aligned} \dot{\alpha}_1 &= e_1(e_2 - e_1) - k_5 e_\alpha, \\ \dot{\gamma}_1 &= e_2^2 - k_6 e_\gamma, \\ \dot{\beta}_1 &= -e_3^2 - k_7 e_\beta, \\ \dot{\eta}_1 &= -e_4^2 - k_8 e_\eta \end{aligned} \quad (12)$$

Where, k_i $(i = 5, 6, 7, 8)$ are positive constants.

By substituting Eqn. (12) into Eqn. (11), one readily obtains

$$\begin{aligned} \dot{V} &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_{\alpha^2} \\ &\quad -k_6 e_{\gamma^2} - k_7 e_{\beta^2} - k_8 e_\eta \end{aligned} \quad (13)$$

which is a negative definite function on $\Re^8$

Hence, according to Lyapunov stability theory [21], it is obvious that the synchronization error and parameter estimation error decay to the equilibrium point exponentially with time. Therefore, we have proved the following result.

Theorem 2.1: The identical L$\ddot{u}$ systems (1) and (2) with fully unknown parameters will globally synchronized by the adaptive control law (5), where the update law for the parameter estimates are given by (12).

2.2. Simulation Results

In what follows, we examine the effectiveness of the proposed approach via numerical simulation. By utilizing fourth order Runge-Kutta routine with time step 0.001, we solve systems (1) and (2). Numerical simulation was carried out using the following parameter settings: $\alpha = 36.0$, $\beta = 3.0$, $\gamma = 20.0$ and $\eta = 1.0$. The initial conditions of the two systems are freely chosen as follows:
$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.1, 0.1, 0.1, 0.1)$ and
$(y_1(0), y_2(0), y_3(0), y_4(0)) = (-9.9, -4.9, 5.1, 10.1)$
The synchronization error between systems (1) and (2) are depicted in Fig. 2 while Fig. 3 shows that the estimated values of the parameters α_1, β_1, γ_1 andη_1 converge to the system parameters $\alpha = 36.0$, $\beta = 3.0$, $\gamma = 20.0$ and$\eta = 1.0$, respectively. The controllers are activated at $t \geq 20$.

3. Adaptive Synchronization of Identical LS Systems

In this section, we present adaptive synchronization of identical hyperchaotic LS system.

3.1. Controller design

Here, some adaptive control laws and parameter update laws for synchronization of identical LS systems [14] are presented, where the parameters of both the drive and response systems are unknown.

Let us consider a drive hyperchaotic LS system of the form [14]:

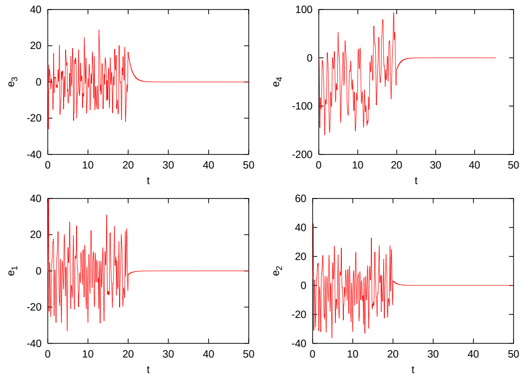

FIG. 2: Synchronization dynamics of identical hyperchaotic Lü systems. The controllers are activated at $t \geq 20$

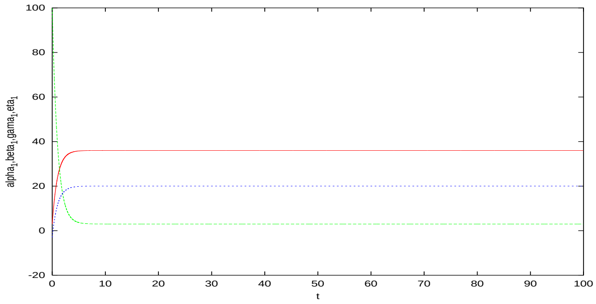

FIG. 3: Parameter estimates of α_1, β_1, γ_1 andη_1

$$
\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + bx_4, \\
\dot{x}_2 &= x_1(c - x_3) - x_2, \\
\dot{x}_3 &= x_1 x_2 - dx_3, \\
\dot{x}_4 &= -x_1 - ax_4
\end{aligned} \tag{14}
$$

In Eqn. (14) x_i $(i = 1, 2, 3, 4)$ are the state variables and a, b, c, and d are the unknown parameters of the system. The state orbits of LS system - depicted in Fig. 4 - is hyperchaotic when the parameter values are $a = 1.0$, $b = 1.5$, $c = 26.0$, and $d = 0.7$

As the responding system, let us consider the controlled LS [14] dynamics described by

$$
\begin{aligned}
\dot{y}_1 &= a(y_2 - y_1) + by_4 + u_1, \\
\dot{y}_2 &= y_1(c - y_3) - y_2 + u_2, \\
\dot{y}_3 &= y_1 y_2 - dy_3 + u_3, \\
\dot{y}_4 &= -y_1 - ay_4 + u_4
\end{aligned} \tag{15}
$$

Where, y_i (i=1,2,3,4) are the state variables and u_i are the nonlinear controllers to be derived.

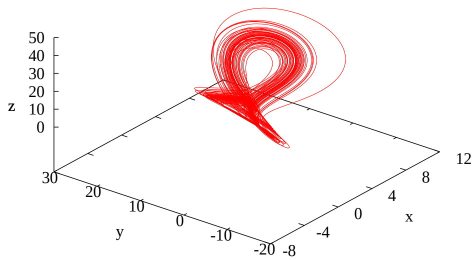

FIG. 4: Phase portrait of the Chaotic attractor of the hyperchaotic LS system with the following parameters $a = 1.0$, $b = 1.5.0$, $c = 26.0$ and $d = 0.7$.

Let us define the synchronization error, e, as follows

$$
e = y_i - x_i, \quad (i = 1, 2, 3, 4). \tag{16}
$$

By subtracting Eqn. (14) from Eqn. (15) and applying the definition of error system in (16), one readily obtains a time varying error system:

$$
\begin{aligned}
\dot{e}_1 &= a(e_1 - e_2) - be_4 + u_1, \\
\dot{e}_2 &= ce_1 - e_2 - y_1 e_3 - x_3 e_1 + u_2, \\
\dot{e}_3 &= y_2 e_1 + x_1 e_2 - de_3 + u_3, \\
\dot{e}_4 &= -e_1 - ae_4 + u_4
\end{aligned} \tag{17}
$$

Let us define the adaptive control functions u_i, $(i = 1, 2, 3, 4)$ as follows:

$$
\begin{aligned}
u_1 &= a_1(e_2 - e_1) + b_1 e_4 - k_1 e_1, \\
u_2 &= -c_1 e_1 + e_2 + y_1 e_3 + x_3 e_1 - k_2 e_2, \\
u_3 &= -y_2 e_1 - x_1 e_2 + d_1 e_3 - k_3 e_3, \\
u_4 &= e_1 + a_1 e_4 - k_4 e_4
\end{aligned} \tag{18}
$$

Where, k_i $(i = 1, 2, 3, 4)$ are positive constants and a_1, b_1, c_1 and d_1 are estimates of a b, c and d, respectively.

By substituting Eqn. (18) into Eqn. (17), we obtain

$$
\begin{aligned}
\dot{e}_1 &= (a - a_1)(e_1 - e_2) - (b - b_1)e_4 - k_1 e_1, \\
\dot{e}_2 &= (c - c_1)e_1 - k_2 e_2, \\
\dot{e}_3 &= -(d - d_1)e_3 - k_3 e_3, \\
\dot{e}_4 &= -(a - a_1)e_4 - k_4 e_4
\end{aligned} \tag{19}
$$

We define the parameter errors as:

$$
e_a = a - a_1, \quad e_b = b - b_1, \quad e_c = c - c_1, \quad e_d = d - d_1 \tag{20}
$$

Substituting Eqn. (20) into Eqn. (19), the error dynamics become

$$\begin{aligned} \dot{e}_1 &= e_a(e_1 - e_2) - e_b e_4 - k_1 e_1, \\ \dot{e}_2 &= e_c e_1 - k_2 e_2, \\ \dot{e}_3 &= -e_d e_3 - k_3 e_3, \\ \dot{e}_4 &= -e_a e_4 - k_4 e_4 \end{aligned} \quad (21)$$

From Eqn. (20) we note that

$$\dot{e}_a = -\dot{a}_1, \quad \dot{e}_b = -\dot{b}_1, \quad \dot{e}_c = -\dot{c}_1, \quad \dot{e}_d = -\dot{d}_1 \quad (22)$$

We utilize Lyapunov stability theorem to derive the update law for adjusting the estimates of the parameters. For this purpose, let us assume a positive definite quadratic Lyapunov function V of the form

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2) \quad (23)$$

Differentiating system (23) along with the error trajectories (21) and using Eqn. (22), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad (24)$$

Where, $\alpha_1 = e_a[e_1(e_1 - e_2) - e_4 e_4 - \dot{a}_1]$, $\alpha_2 = e_b[-e_1 e_4 - \dot{b}_1]$, $\alpha_3 = e_c[e_1 e_2 - \dot{c}_1]$, and $\alpha_4 = e_d[e_3 e_3 - \dot{d}_1]$.

In view of Eqn. (24), the estimated parameters are updated by the following laws:

$$\begin{aligned} \dot{a}_1 &= e_1(e_1 - e_2) - e_4 e_4 - k_5 e_a, \\ \dot{b}_1 &= -e_1 e_4 - k_6 e_b, \\ \dot{c}_1 &= e_1 e_2 - k_7 e_c, \\ \dot{d}_1 &= e_3 e_3 - k_8 e_d \end{aligned} \quad (25)$$

Where, k_i $(i = 5, 6, 7, 8)$ are positive constants.

Using Eqn. (25) in (24), the derivative of the Lyapunov function becomes

$$\begin{aligned} \dot{V} = & -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \\ & -k_5 e_a^2 - k_6 e_b^2 - k_7 e_c^2 - k_8 e_d^2 \end{aligned} \quad (26)$$

which is a negative definite function on $\Re^8$

Hence, based on Lyapunov stability theory [21], it is clear that the synchronization error and parameter estimation error decay to the origin exponentially with time. Therefore, we have proved the following important result.

Theorem 3.1: The identical LS systems (14) and (15) with unknown parameters will globally synchronize by the adaptive control law (18), where the update law for the parameter estimates are given by (25).

3.2. Simulation results

Here, we validate the above theoretical analysis via numerical simulation. Numerical experiments are carried out using fourth order Runge-Kutta algorithm with time step 0.001 to solve systems (14) and (15). We carried out numerical simulations using the following parameter settings: $a = 1.0$, $b = \frac{3}{2}$, $c = 26.0$ and $d = 0.7$. The initial conditions of the two systems are freely chosen as follows:
$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.3, 3.5, 4.2, 1.2)$ and
$(y_1(0), y_2(0), y_3(0), y_4(0)) = (1.0, -1.0, 1.0, 0.0)$
The synchronization error between systems (14) and (15) are depicted in Fig. 5 while Fig. 6 shows that the estimated values of the parameters a_1, b_1, c_1 and d_1 converge to the system parameters $a = 1.0$, $b = 1.5$, $c = 26.0$ and $d = 0.7$ respectively. The controllers are activated at $t \geq 20$.

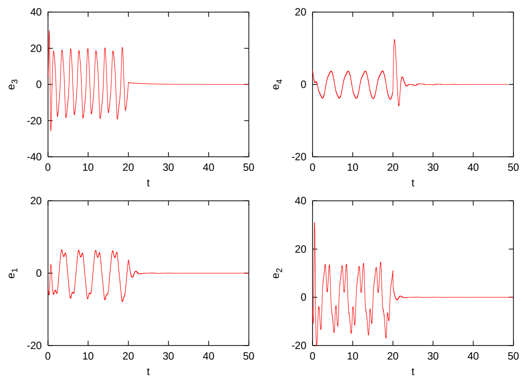

FIG. 5: Synchronization dynamics of identical hyperchaotic LS systems. The controllers are activated at $t \geq 20$.

4. Adaptive Synchronization of Non-identical Hyperchaotic Lü and LS Systems

In what follows, we present adaptive synchronization of non- identical hyperchaotic Lü and LS systems.

4.1. Controller design

Some adaptive control laws and parameter update laws for synchronization of non- identical Lü and

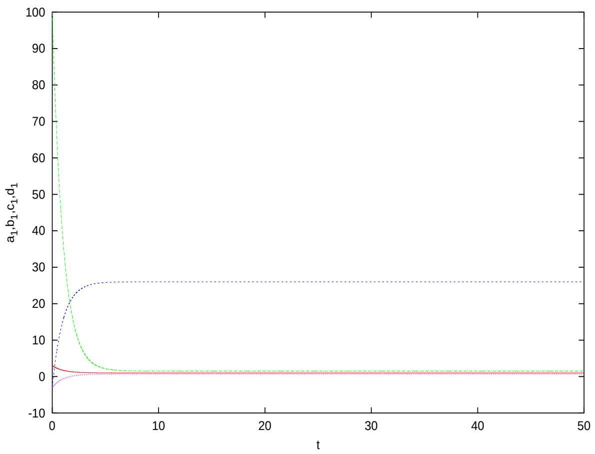

FIG. 6: Parameter estimates of a_1, b_1, c_1 and d_1.

LS systems are derived in this section. The hyperchaotic L$\ddot{u}$ system [20] with known parameter is the drive system while the response system consist of LS [14] system whose parameters are assumed to be unknown.

Let us consider drive hyperchaotic L$\ddot{u}$ system of the form [20]:

$$\begin{aligned} \dot{x}_1 &= -\alpha(x_1 - x_2) + x_4, \\ \dot{x}_2 &= \gamma x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - \beta x_3, \\ \dot{x}_4 &= x_1 x_3 + \eta x_4 \end{aligned} \tag{27}$$

Where, x_i (i=1,2,3,4) are the state variables and α, β, γ and η are the parameters of the system.

The response system consists of hyperchaotic LS [14] dynamics described by

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + by_4 + u_1, \\ \dot{y}_2 &= y_1(c - y_3) - y_2 + u_2, \\ \dot{y}_3 &= y_1 y_2 - dy_3 + u_3, \\ \dot{y}_4 &= -y_1 - ay_4 + u_4 \end{aligned} \tag{28}$$

Where, y_i $(i = 1, 2, 3, 4)$ are the state variables, a, b, c and d are the unknown parameters of the system; and u_i are the nonlinear controllers to be derived.

Let us define the synchronization error, e, as follows

$$e = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{29}$$

By subtracting Eqn. (27) from Eqn. (28) and applying the definition of error system in system (29), one readily obtains a time varying error system:

$$\begin{aligned} \dot{e}_1 &= a(y_2 - y_1) + by_4 + \alpha(x_1 - x_2) - x_4 + u_1, \\ \dot{e}_2 &= y_1(c - y_3) - y_2 - \gamma x_2 + x_1 x_3 + u2, \\ \dot{e}_3 &= y_1 y_2 - dy_3 - x_1 x_2 + \beta x_3 + u3, \\ \dot{e}_4 &= -y_1 - ay_4 - x_1 x_3 - \eta x_4 + u4 \end{aligned} \tag{30}$$

Let the adaptive control functions u_i, $(i = 1, 2, 3, 4)$ be defined as follows:

$$\begin{aligned} u_1 &= -a_1(y_2 - y_1) - b_1 y_4 - \alpha(x_1 - x_2) + x_4 - k_1 e_1, \\ u_2 &= -y_1(c_1 - y_3) + y_2 + \gamma x_2 - x_1 x_3 - k_2 e_2, \\ u_3 &= -y_1 y_2 + d_1 y_3 + x_1 x_2 - \beta x_3 - k_3 e_3, \\ u_4 &= y_1 + a_1 y_4 + x_1 x_3 + \eta x_4 - k_4 e_4 \end{aligned} \tag{31}$$

Where, k_i $(i = 1, 2, 3, 4)$ are positive constants and a_1, b_1, c_1 and d_1 are estimates of a b, c and d, respectively.

By substituting Eqn. (31) into Eqn. (30), we obtain

$$\begin{aligned} \dot{e}_1 &= (a - a_1)(y_2 - y_1) + (b - b_1)y_4 - k_1 e_1, \\ \dot{e}_2 &= (c - c_1)y_1 - k_2 e_2, \\ \dot{e}_3 &= -(d - d_1)y_3 - k_3 e_3, \\ \dot{e}_4 &= -(a - a_1)y_4 - k_4 e_4 \end{aligned} \tag{32}$$

We define the parameter errors as:

$$e_a = a - a_1, \quad e_b = b - b_1, \quad e_c = c - c_1, \quad e_d = d - d_1 \tag{33}$$

Substituting Eqn. (33) into Eqn. (32), the error dynamics become

$$\begin{aligned} \dot{e}_1 &= e_a(y_2 - y_1) + e_b y_4 - k_1 e_1, \\ \dot{e}_2 &= e_c y_1 - k_2 e_2, \\ \dot{e}_3 &= -e_d y_3 - k_3 e_3, \\ \dot{e}_4 &= -e_a y_4 - k_4 e_4 \end{aligned} \tag{34}$$

From Eqn. (33) we note that

$$\begin{aligned} \dot{e}_a &= -\dot{a}_1, \\ \dot{e}_b &= -\dot{b}_1, \\ \dot{e}_c &= -\dot{c}_1, \\ \dot{e}_d &= -\dot{d}_1 \end{aligned} \tag{35}$$

We utilize Lyapunov stability theorem to derive the update law for adjusting the estimates of the parameters. For this purpose, let us assume a positive definite quadratic Lyapunov function V of the form

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2) \tag{36}$$

Differentiating Eqn. (36) along with the error trajectories (34) and using Eqn. (35), we obtain

$$\begin{aligned}\dot{V} &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \\ &\quad +\beta_1 + \beta_2 + \beta_3 + \beta_4 \qquad (37)\end{aligned}$$

Where, $\beta_1 = e_a[e_1(y_2 - y_1) - e_4 y_4 - \dot{a}_1]$, $\beta_2 = e_c[y_1 e_2 - \dot{c}_1]$, $\beta_3 = e_b[e_1 y_4 - \dot{b}_1]$, and $\beta_4 = e_d[-e_3 y_3 - \dot{d}_1]$.
In view of Eqn. (37), the estimated parameters are updated by the following laws:

$$\begin{aligned}\dot{a}_1 &= e_1(y_2 - y_1) - e_4 y_4 - k_5 e_a, \\ \dot{b}_1 &= e_1 y_4 - k_6 e_b, \\ \dot{c}_1 &= e_2 y_1 - k_7 e_c, \\ \dot{d}_1 &= -e_3 y_3 - k_8 e_d \qquad (38)\end{aligned}$$

Where, k_i $(i = 5, 6, 7, 8)$ are positive constants.
Using Eqn. (38) in (37), the derivative of the Lyapunov function becomes

$$\begin{aligned}\dot{V} &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_a^2 \\ &\quad -k_6 e_b^2 - k_7 e_c^2 - k_8 e_d^2 \qquad (39)\end{aligned}$$

which is a negative definite function on $\Re^8$
Hence, based on Lyapunov stability theory[21], it is obvious that the synchronization error and parameter estimation error decay to the origin exponentially with time. Therefore, we have proved the following important result.

Theorem 4.1: The non-identical systems (27) with known parameters and (28) with unknown parameters will globally and exponentially synchronize by the adaptive control law (31), where the update law for the parameter estimates are given by (38).

4.2. Simulation results

Here, we carry out numerical experiment using fourth order Runge-Kutta algorithm with time step 0.001 to solve systems (27) and (28). We carried out numerical simulations using the following parameter settings: $a = 1.0$, $b = \frac{3}{2}$, $c = 26.0$ and $d = 0.7$. The initial conditions of the two systems are freely chosen as follows:
$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.1, 0.5, 0.2, 1.2)$ and
$(y_1(0), y_2(0), y_3(0), y_4(0)) = (1.5, -1.3, 1, 0.5)$
The synchronization error between systems (27) and (28) are depicted in Fig. 7 while Fig. 8 shows that the estimated values of the parameters $a = 1.0$, $b = \frac{3}{2}$, $c = 26.0$ and $d = 0.7$ converge to the system parameters $a = 1.0$, $b = \frac{3}{2}$, $c = 26.0$ and $d = 0.7$, respectively. The controllers are activated at $t \geq 20$.

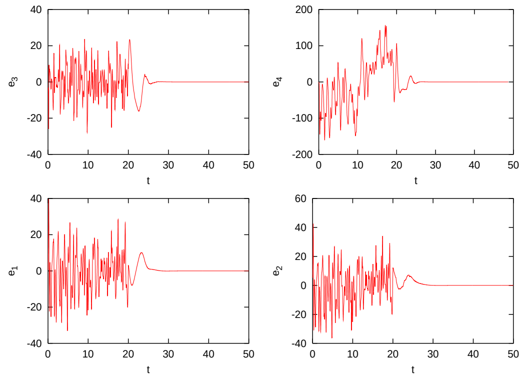

FIG. 7: Synchronization dynamics of hyperchaotic Lü and LS systems. The controllers are activated at $t \geq 20$

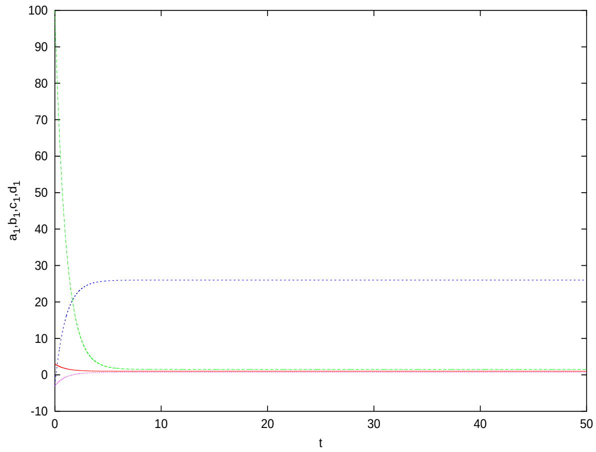

FIG. 8: Parameter estimates of a_1, b_1, c_1 and d_1 for the non-identical Lü and LS systems.

5. Summary and Conclusion

In summary, nonlinear controllers have been derived for the synchronization of two identical and non-identical hyperchaotic systems with unknown parameters. Based on the Lyapunov stability theorem, the adaptive control laws for synchronization controllers associated with adaptive update laws of system parameters were devised to make the states of two identical and non-identical systems asymptotically synchronized. The correctness of the obtained controller has been demonstrated. The numerical simulation further shows the effectiveness of the derived controllers.

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