On Hexagonal Geometries in Physics and Lie Algebras

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The connection between physics and geometry has led to a better understanding of many open questions in physics and mathematics. Many of such questions have been explored in the context of quantum field theories, string theory, quantum gravity, and particle physics. In this paper, we are interested in hexagonal geometries appearing in rank two root systems of Lie algebra structures. In particular, we show that such geometries appear in many areas of physics. This includes strings, M-theory, nanotechnology materials and network systems.

1. Introduction

Geometry is one of the most fundamental concepts not only in mathematics but in physics as well. On many occasions, new physics evolved from geometry. The latter has been considered as a natural instrument not only in higher energy physics and condensed matter, but also in technological science including nanoscale physics and network systems.

The connection between physics and geometry led to a deeper understanding of many physical and mathematical open problems. Many of these problems were explored and theories were developed in the context of quantum filed theories, string theory, quantum gravity, and particle physics.

Hexagonal shape is a nice example of such geometrical entity. Various interesting applications of such a geometry have been developed and have been used in modern physics.

In this work, we are interested in this special geometry. We show that it appears in many areas of physics, including the string theory, nanotechnology material science and network systems. These connections are based on hexagonal geometry appearing in the context of rank two root systems of Lie algebra structures.

2. Lie Algebras

Before showing the existence of the hexagonal physics, let us first recall some basic facts on root system of Lie algebras. Indeed, Lie algebra g is a vector space together with an antisymmetric bilinear bracket defined on it. This bracket, called the Lie bracket, is written as

$$[,]: g \times g \to g \tag{1}$$

and it satisfies the following relation $[a,[b,c]] + [c,[a,b]] + [b,[c,a]] = 0$. This relation is called the Jacobi identity. It is recalled that g is called simple if its only ideals are itself and 0. Consider a semisimple Lie algebra, which is decomposable to a direct product of simple Lie algebras. A toric subalgebra is generated by some semisimple elements of g. The maximal toric subalgebra H is generated by all the semisimple elements. It is worth noting that g may then be written as the direct sum of H and the subspaces g_α

$$g = H \oplus \{\oplus_\alpha g_\alpha\} \tag{2}$$

Where, $\{g_\alpha = x \in g | [h,x] = \alpha(x)x\}$ for $x \in g$. Here α ranges over all elements of the dual of H. They are called roots.

According to [1,2,3], a root system Δ is defined as a subset of the Euclidean space E satisfying the following constraints:

1. Δ is finite and spans E, $0 \notin \Delta$
2. If $\alpha \in \Delta$, $k\alpha \in \Delta$ id $k = \pm 1$
3. $\forall \alpha \in \Delta$, σ_v leaves invariant Δ, this means $\sigma_\alpha(\Delta) \subset \Delta$
4. if α and β are inside Δ, the quantity $\langle \beta.\alpha \rangle = \frac{2(\beta.\alpha)}{(\alpha.\alpha)} \in Z$.

It has been shown that there is a nice classification of rank two algebras. Indeed, consider two root elements α and β of Δ. It is easy to see that

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4\cos^2\theta$$

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Where, θ is the angle between α and β. This leads to the following constraints

$$0 \ \leq \ \langle \beta, \alpha \rangle \langle \alpha, \beta \rangle \leq 4.$$

It follows that the possible values of θ are $30^0, 45^0, 60^0, 90^0, 120^0, 135^0, 150^0$, leading to a useful classification. More details on this classification can be found in [1,2,3].

3. Hexagonal Geometry in String Theory

In this section, we discuss the hexagonal geometry within Lie algebras used in string theory and related models. A particular emphasis will be on hexagonal root systems corresponding to holonomy groups used in the compactification down to four dimensions.

It is recalled that string theory extends to physics describing the motion of particles (zero dimensional objects) to one relating with one (or higher) dimensional objects [4,5]. It has been considered as a possible unified theory of strong interaction, weak interaction, electromagnetic interaction, and gravitation interaction. In fact, there are two string configurations:

1. Open string theory described by gauge theories living on D-branes.
2. Closed string theory controlling gravity theory.

A deeper study reveals that string theory produces five different models. These models are defined in ten dimensional spacetime with some particular Lie symmetries, including $E_8 \times E_8$ and $SO(32)(D_{16})$, considered as gauge groups. It turns out that, string theory suffers from many problems. However, such problems can be removed partially using the following scenarios:

- Compactification
- Introducing new theories.

This includes M-theory in eleven dimensions proposed by Witten (1995) [6] and F-theory in twelve dimensions proposed by Vafa (1996) [7].

Connecting such models with real world requires compact manifolds with special Lie symmetries appearing as holonomy groups [8,9,10]. For instance, in string theory for $\mathcal{N}=1$ supersymmetry to be in our usual world, we have to compact the remaining $10-4=6$ dimensions in a Calabi-Yau threefold having SU(3) as a holonomy group. However, working with M-theory we need a seven-dimensional $(7=11-4)$ with G_2 holonomy [11].

These two holonomy groups share nice properties. In particular, both groups have an amazing relation connecting the dimension and the rank

$$\text{Dim } g = \text{rank } g + 6 \times m. \tag{3}$$

They also involve a hexagonal geometry in their root systems. To see that, let us consider string theory where the relevant holonomy group is SU(3). In Lie algebraic language, this symmetry has the following properties:

- Dimension: $\dim A_2 = 2 + 6 \times 1$
- Two simple roots α_1, α_2 of equal lengths and at an of 120°

$$|\alpha_1| = |\alpha_2|, \qquad \widehat{(\alpha_1, \alpha_2)} = 120°.$$

- Hexagonal root system:

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\} \tag{4}$$

In the case of M-theory theory, the desired holonomy group is fixed to G_2. This symmetry has the following properties:

- Dimension: $\dim G_2 = 14 = 2 + 6 \times 2$
- Two simple roots: α_1, α_2 of nonequal length, at 150° angle:

$$\frac{|\alpha_2|^2}{|\alpha_1|^2} = 3, \qquad \widehat{(\alpha_1, \alpha_2)} = 120 + 30°.$$

- Two rank root system:

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2),$$

$$\pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$$

It is easy to show that the root systems of G_2 involves two hexagonal geometries. In fact, there are two hexagons of unequal side length at angle 30°. The small hexagon is associated with $\{\pm\alpha_1, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$, while the roots $\{\pm\alpha_2, \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$ are associated with the big hexagon.

In what follows, we will see that the above hexagonal geometries can be explored in many areas including new technology.

4. Hexagonal Geometry in Materials

Recently, there has been a nice attempt to make a contact between high energy physics and graphene physics using AdS/CFT correspondence in string theory [12]. Motivated by this work and hexagonal geometry in string theory compactification, we try to show that the double hexagonal structure can be used in materials including graphene and silicene. It is recalled that hexagonal geometry appears in many places in condensed matter and it can be considered as the most stable geometry explored in the semiconductor electronic applications. Many efforts have been devoted to study the physics of materials having such a geometry using different calculation methods with appropriate approximations. The most studied models are graphene and silicene [13,14,15]. More recently, an unexpected superstructure with hexagonal geometry has been found in silicene [16]. It is associated with particular superstructure given by $(\sqrt{3}\times\sqrt{3})R30^{\circ}$. This new hexagonal geometry can be associated with Lie algebra structure of G_2 and it can be used to build to a new hexagonal material. To understand the connection with G_2, one should first establish the link with the single hexagonal geometry appearing in A_2 Lie algebras discussed in the previous string theory part. Indeed, as we have seen in the previous section, this Lie algebra involves two simple roots of equal length at 120° angle. It turns out that one can make a nice connection between the root system of Lie algebras of rank two and materials exhibiting the hexagonal geometry. To make this contact easy to understand, we consider a single hexagonal unit cell with (1×1) structure. Roughly, the correspondence is presented as follows:

- Each root A_2 is associated with a Si atom placed on hexagonal unit cell.
- The lattice parameter a is associated with the length of the roots.

This nice interplay can be completed with the fact that hexagons can tessellate the full plane forming the supercell crystal structure of silicene.

It is possible to engineer materials with a double hexagonal geometry considered as a possible extension of the single hexagonal materials associated with A_2 Lie algebras. At this level, one can imagine that the exceptional Lie G_2 could produce materials with double hexagonal geometry. The geometry is associated with the root system of G_2. More specifically, to each non zero root we associate a single silicon atom. In this new atomic representation, the principal hexagonal cell consists of 12 atoms instead of six appearing in the usual material. Inspired from the root equation given in the previous section, the unit cell involves two hexagons of unequal side length at an angle of 30°. Based on G_2 Lie algebras, the corresponding lattice parameters, which will be noted here as a_1 and a_2, are associated respectively with the lengths of two simple roots of G_2 Lie algebras. The two lattice parameters are constrained by

$$\frac{|a_2|^2}{|a_1|^2} = 3.$$

required by the G_2 Cartan matrix.

As we have seen in the previous model with single hexagonal geometry, G_2 Lie algebra allows one to build two superstructures given by $a(1\times 1)$ and $a(\sqrt{3}\times\sqrt{3})$ producing materials with double hexagonal geometries on the same sheet materials [17].

5. Hexagonal Geometry in Network Systems

In network systems, the cellular concept plays an important role in solving the problem of spectral congestion and capacity. The cellular concept is obtained by replacing high cell power transmitter (large cell) with many small cells. In this configuration, each cell provides coverage to only a small area. It turns out that the number of cells must be increased to improve the user capacity. To realize a total coverage, the choice of a structure that can be overlaid upon a map without leaving gaps or creating overlapping area is necessary. In network systems, the hexagonal geometry has been used to model coverage due to its semi-realistic behavior [18].

As we have seen before one may use hexagonal root systems of Lie algebra to bring new hexagonal geometry in network systems. Roughly speaking, we will see that the root system can be associated with radio system in network systems. In fact, a unit hexagonal cell in telecom systems is associated with the six nonzero vector roots of A_2 Lie algebra. Indeed, each root is associated with a coverage limit. For A_2 algebra, all roots have equal length, which is associated with the coverage radius. Since the base station is placed at the center of each hexagonal cell, it should be associated with the zero roots corresponding to the Cartan subalgebra.

The full picture of network system can be obtained by using the fact that A_2 hexagons tessellate the full plane making a hexagonal cellular shape for network systems. As we have seen before G_2 contains a special hexagonal geometry. In particular, it involves two hexagons of unequal side length generated by two simple unequal roots. As in the case of A_2 hexagons, the converge limits should be associated with the root system of G_2. More precisely, each unit cell consists of 12 coverage limits instead of six appearing in single hexagonal geometry.

Roughly speaking, a close inspection shows that one can propose a nice correspondence between the rank 2 root systems and cellular network systems. Roughly, we summarize the mapping in the following table:

Rank two Lie algebras	Cellular network systems
Root systems	Radio systems
Non zero roots	coverage limiting
zero roots	base stations
Simple roots	frequency bands
Simple roots with equal length	one frequency band
Simple roots with unequal length	two frequency bands

In fact, the implementation of the double hexagonal geometry in cellular network systems can be supported by the existence of two frequency bands. Each frequency band can be associated with a single hexagonal structure. In this G_2 hexagonal geometry, the frequency bands can be collocated on the same area in contrast to the single hexagonal geometry where it can appear only as one frequency band without sharing the same base station. It is possible that this double hexagonal geometry can bring novelties in higher density population and missing space room problems leading to some improvements [19].

6. Discussions

It has been shown that hexagonal geometry appears in many areas of physics. Inspired by the hexagonal geometry developed in rank two root systems of Lie algebra structures, we have shown the role played by such a geometry in string theory, condensed matter and many other physics. A special focus has been on A_2 and G_2 hexagons.

This work comes up with many questions related to hexagonal geometry. A curious question is related with the work on string realization of graphene [12]. It should be interesting to make contact with such a work based on the fact that G_2 manifolds can be used to get three dimensional field theories from string theory.

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