On Non-Commutative Black Holes and Their Thermodynamics in Arbitrary Dimension

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Motivated by new developments in string theory, we investigate black hole solutions both analytically and numerically as well as their thermodynamics on the non-commutative space with arbitrary dimensions. Specifically, we first compute analytically the horizons of extremal geometries of various black hole types. Then, we numerically analyze the thermodynamical properties of such non-commutative black holes, including the Hawking temperature and the entropy functions. The effect of non-commutative space on such parameters is discussed. These calculations reveal many deviations from the ordinary solutions. For small values of θ, we show that there exists a linear relation between the maximal temperature and closed string background.

1. Introduction

Four dimensional Reissner-Nordstrom black holes are static and spherically symmetric configurations, which are known to minimize the Maxwell-Einstein action. The solution of such black holes is completely defined by two parameters: the charge Q and the mass M. The generalization to arbitrary dimensional space time has been given using the recent developments in string theory and related topics [1-4].

Such black holes exhibit an interesting phenomenon called the attractor mechanism. At the horizon, the moduli scalar fields take fixed values given in terms of the black hole charges. This mechanism can be understood in terms of an effective potential, which depends on the charges and the moduli coming from string theory compactifications [5]. Extremizing the potential with respect to the moduli, the minimum gives such fixed values. In five dimensions, for instance, an M-theory realization of such black hole attractors has been studied in [6]. Moreover, the black attractors in six, seven and eight dimensions have been explored in [7]. In particular, the extremal BPS and non BPS black attractors in $N = 2$ supergravity, embedded in type IIA superstring theory and M-theory on the K3 surface, respectively, have been studied extensively in literature. The attractor mechanism equations and their solutions have been treated using the criticality condition of the attractor potential of such black objects.

On the other hand, black hole solutions on the non-commutative geometry have been subject to increasing interest in connection with higher dimensional theories too [8-12]. In particular, the non-commutative black holes are involved in the study of string and M-theory, particularly in the Matrix model construction [8,9].

In string theory, the non-commutative geometry appears naturally from the study of the open string theories interacting with closed string background fields. The boundary conditions for the open strings living on D-branes depend on the values of the NS-NS B-field [13]. In Seiberg-Witten limit, the correlation functions of one-dimensional boundary string fields $x^\mu(t)$ obey nontrivial commutation relations leading to a non-commutative space-time. In this way, the usual product of functions is replaced by the star product of Moyal Bracket [14-20]. Based on these results, several works have been devoted to build the quantum field theory on non-commutative spaces, including various black hole solutions [21-28]. Up to the second order of perturbative calculations, the event horizon of a four-dimensional Schwarzschild black hole is derived for non-commutative geometries [23]. It has been found that the event horizon in such geometries is less than the event horizon in commutative spaces.

The aim of this paper is to contribute to these topics by studying analytically and numerically

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non-commutative black hole solutions in arbitrary dimensions. This work is motivated by the standard non-commutative analysis in field theories showing new parameters, which could be used in the framework of the moduli stabilization mechanism in string theory compactification. Specifically, we study the analytic gravitational solution of spherically symmetric objects in the presence of extra physical constants including cosmological one. In particular, we present here exact equations of motion and horizon geometries of various black hole solutions extending the previously mentioned works. Thermodynamical properties of such non-commutative black holes, including the Hawking temperature and the entropy, are also computed and discussed numerically.

2. Non-commutative Schwarzschild Black Holes

Motivated by string theory results on black objects, we discuss black hole solutions in various dimensions. Then we interpret the results obtained and make a contact with the commutative case given in literature. It turns out that there are many ways to introduce the non-commutative geometry in the black hole background. Here we use the non-commutative variables subject to the following non-commutative commutation relations [21]

$$[\widehat{x}_\mu, \widehat{x}_\nu] = \widehat{x}_\mu \star \widehat{x}_\nu - \widehat{x}_\nu \star \widehat{x}_\mu \quad (1)$$

$$= i\theta_{\mu\nu} \quad (2)$$

Where, the non-commutative parameter $\theta_{\mu\nu}$ possesses the dimension of $(\text{length})^2$. It has been shown in [13] that the above commutation relations have a nice interpretation in terms of open string interacting with closed string background fields g_{ij} and B_{ij} using Seiberg-Witten method.

In the non-commutative geometry context, the usual functional product should be replaced by the star-product defined by

$$(f \star g) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x)g(y)|_{x=y} \quad (3)$$

Where, f and g are two arbitrary functions assumed to be infinitely differentiable.

In what follows, we study the black holes solutions in non-commutative space with arbitrary dimensions. To start, we consider the simplest case of black holes known by Schwarzschild solutions in D dimensions [23]. This solution minimizes the following action

$$S = \frac{1}{16\pi G_D}\int d^D x\sqrt{-g}R \quad (4)$$

Where, G_D is the gravitational constant, R the Ricci scalar. Varying this action with respect to the metric tensor leads to the well-known Schwarzschild metric

$$ds^2 = -\left(1-\left(\frac{r_H}{r}\right)^{D-3}\right)dt^2 + \frac{dr^2}{1-\left(\frac{r_H}{r}\right)^{D-3}} + r^2 d\Omega^2_{D-2} \quad (5)$$

Where, $r_H^{D-3} = \frac{16\pi M G_D}{(D-2)\Omega_{D-2}}$ and where $\Omega_D = \frac{2\pi^{(D+1)/2}}{\Gamma\left(\frac{D+1}{2}\right)}$. It is worth noting that for $D = 4$, this equation can be reduced to $r_H = 2MG$, which is the radius of the horizon in four dimensions.[2].

As in [23], we shalll consider the proposed metric solution on the non-commutative space

$$ds^2 = -\left(1-\left(\frac{r_H}{\sqrt{\hat{r}\hat{r}}}\right)^{D-3}\right)dt^2 + \frac{d\hat{r}d\hat{r}}{\left(1-\left(\frac{r_H}{\sqrt{\hat{r}\hat{r}}}\right)^{D-3}\right)} + \hat{r}\hat{r}d\Omega^2_{D-2} \quad (6)$$

Where, now the horizon $\hat{r}$ should satisfy the following constraint

$$1-\left(\frac{r_H}{\sqrt{\hat{r}\hat{r}}}\right)^{D-3} = 0 \quad (7)$$

In this formalism, a new coordinate system should be used

$$x_i = \hat{x}_i + \frac{1}{2}\theta_{ij}\hat{p}_j, \quad p_i = \hat{p}_i \quad (8)$$

It is recalled that the new variables are shown to satisfy the usual canonical commutation relations:

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[2] With the use of the conventions $\hbar = c = k_B = \frac{1}{4\pi\epsilon_0} = 1$ and $G = m_{Pl}^{-2} = \ell_{Pl}^{-2}$ where m_{P_l} and ℓ_{P_l} are the Planck mass and the Planck length, respectively.

$$[x_i, x_j] = 0, \quad [x_i, p_j] = i\delta_{ij}, \quad [p_i, p_j] = 0 \qquad (9)$$

As in [23], changing the variables $\hat{x}$ to x in (2.7), the singularities of the metric (2.6) are solutions of

$$1 - \left(\frac{r_H}{\sqrt{(x_i - \theta_{ij}p_j/2)(x_i - \theta_{ik}p_k/2)}} \right)^{(D-3)} = 0 \qquad (10)$$

This can be expanded as

$$1 - \left(\frac{r_H}{r}\right)^{D-3} \left[1 + \left(\frac{D-3}{2}\right) \frac{x_i \theta_{ij} p_j}{r^3} + \left(\frac{D-3}{8}\right) \frac{\theta_{ij}\theta_{ik}p_j p_k}{r^2} \right] + \mathcal{O}(\theta^3) + \ldots = 0 \qquad (11)$$

In what follows, we take a particular case corresponding to $\theta = \theta_\xi$, and the remaining θ components equal to zero. Then we consider a special gauge given by $x_i\theta_{ij}p_j = 0$, the above equation reduces to

$$r^{D-1} - r_H^{D-3} r^2 + \frac{r_H^{D-3}}{32}(D-3)(\sum_{i \neq \xi}^{D-1} p_i^2)\theta^2 + \mathcal{O}(\theta^3) + \ldots = 0 \qquad (12)$$

Where, $\theta^2 = \theta_i\theta_i$. This equation can be considered as $(D-1)$-polynomial equation with the following form

$$r^{D-1} + ar^2 + b = 0 \qquad (13)$$

Where, $a = -r_H^{D-3}$ and $b = \frac{r_H^{D-3}}{32}(D-3)\sum_{i\neq\xi}^{D-1} p_i^2\theta^2$. The solution of this equation will give the horizon radius in D-dimensional non-commutative space. Performing algebraic straightforward computations, we can derive explicit solutions up to $D = 5$. It is worth nothing that for $D = 4$, we can recover the results presented in literature [23]. However, $D = 5$ comes up with new results. Based on the Eqn. (13), we obtain two kind of physical solutions, namely

$$\hat{r}_{h\pm} = \frac{r_H}{\sqrt{2}} \sqrt{1 \pm \sqrt{1 - \frac{1}{4}\left(\frac{\theta}{r_H}\right)^2 \sum_{i\neq\xi}^{D-1} p_i^2}} \qquad (14)$$

At this level, some remarks should be given

- The key point concerns the standard limit which gives $\hat{r}_{h_+} = r_H$ once the deformation parameter θ is set equal to zero.

- For higher dimensions ($D > 5$), there is no technical way, to our knowledge, to solve analytically the above Eqn. (13). However, it is possible to use numerical techniques to get some explicit solutions.

Having discussed the effect of the non commutativity of space on the radius of the horizon, we will extend these results to others black holes involving extra parameters including charges.

3. Non-commutative Reissner Nordstrom Black Holes

In this section, we consider the case of the Reissner Nordstrom black hole being a generalization of Schwarzschild black hole. As mentioned before, Reissner-Nordstrom black holes are static and spherically symmetric configurations which are known to minimize the Maxwell-Einstein action. The solution in four dimensions was first found in [29,30]. Every solution of this kind is completely defined by giving two parameters: the charge Q and the mass M. The generalization to higher dimensional space-times with a cosmological constant was given by Tangherlini in [31,32]. For certain range of parameters, the geometry of these objects present three horizons: Cauchy, black hole and cosmological. The thermodynamical properties of black holes permits those system to dynamically vary some parameters of the moduli space. For instance, the evaporation process may reduce the mass of a charge black hole to the point of coalescence at which the two inner horizons lead to a degenerate solution called the extreme black hole.

Reissner-Nordstrom black holes in higher dimensions are also known to be static and spherically symmetric configurations minimizing the Einstein-Hilbert action

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \qquad (15)$$

in a D spacetime. The field strength F is a closed 2-form, where it can always be locally written as $F = dA$, where A is the potential 1-form. Indeed, the variation of this action with respect to the metric tensor $g_{\mu\nu}$ leads to the well-known Reissner Nordstrom metric which can be written as follows

$$ds^2 = -\left(1 - \frac{2\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}\right) dt^2 + \left(1 - \frac{2\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}\right)^{-1} dr^2 + d\Omega^2_{D-2} \quad (16)$$

Where, $d\Omega^2_{D-2}$ is the line element on the (D-2)-dimensional unit sphere and D is space time dimensionality. The volume of the $(D-2)$-dimensional unit sphere is given by

$$\Omega^2_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}. \quad (17)$$

The element of the metric g_{00} is a function of the mass and the charge given in terms of parameters μ and q as follows

$$\mu = \frac{8\pi G_D}{(D-2)\Omega_{(D-2)}} M, \qquad q = \sqrt{\frac{8\pi G_D}{(D-2)(D-3)}}\, Q \quad (18)$$

G_D is gravitational constant in D-dimensional space-time

$$G_D = \frac{(2\pi)^{D-4}}{\Omega_{D-2}} m_{Pl}^{2-d} \quad (19)$$

Where, m_{Pl} is the D-dimensional Planck mass.

In what follow, we introduce the effect of the non commutativity in the Reissner Nordstrom black hole in arbitrary dimension. Indeed, the horizon of such black holes in non-commutative spaces can be obtained as usual by considering,

$$\hat{g_{00}} = 0 \rightarrow 1 - \frac{2\mu}{(\sqrt{\hat{r}\hat{r}})^{D-3}} + \frac{q^2}{\hat{r}^{2(D-3)}} = 0 \quad (20)$$

Where, $\hat{r}$ now solve (20) in the non-commutative framework. Using similar calculation, we obtain the following equations

$$1 - \frac{2\mu}{r^{(D-3)}} + 2\mu\left(\frac{D-3}{32}\right)\frac{(p^2 - p_\xi^2) f\theta^2}{r^{(D-1)}} + \frac{q^2}{r^{2(D-3)}} - \left(\frac{D-3}{16}\right) q^2\, \frac{(p^2 - p_\xi^2)\theta^2}{r^{2(D-2)}} + \mathcal{O}(\theta^3) + \ldots = 0 \quad (21)$$

This equation can be put under the following polynomial form as follows,

$$r^{2(D-2)} + ar^{D-1} + b^{D-3} + cr^2 + d = 0 \quad (22)$$

Where

$$a = 2\mu, \quad b = 2\mu\left(\frac{D-3}{32}\right)\sum_{i\neq\xi} p_i^2\theta^2, \qquad c = q^2, \quad d = q^2\left(\frac{D-3}{16}\right)\sum_{i\neq\xi} p_i^2\theta^2 \quad (23)$$

It is worth noting that $D = 4$ is obvious and the solution are given in [19]. However, in five dimensional non-commutative space, Eqn. (12) can be written as

$$r^6 + ar^4 + (b+c)r^2 + d = 0 \quad (24)$$

Real positive solutions are given by

$$r_h = \frac{\sqrt{\frac{\sqrt[3]{2}(2a^2+3a\alpha-6c)}{\sqrt[3]{X}} - 2a + 2^{2/3}\sqrt[3]{X}}}{\sqrt{6}} \quad (25)$$

Where

$$X = -2a^3 + 9a\left(c - \frac{a\alpha}{2}\right) - 27c\alpha + \sqrt{\left(2a^3 + \frac{9a^2\alpha}{2} - 9ac + 27c\alpha\right)^2 - \frac{1}{2}\left(2a^2 + 3a\alpha - 6c\right)^3} \quad (26)$$

and

$$\alpha = \left(\frac{D-3}{16}\right)\sum_{i\neq\xi} p_i^2\theta^2 \quad (27)$$

At the end of this section, we note that the above Reissner-Nordstrom black hole calculation can be extended to many geometries. A possible solution is associated with de Sitter-Reissner-Nordstrom black holes in higher dimensions. These solutions are also static and spherically symmetric configurations like Reissner-Nordstrom black hole.

$$S = \int d^D x\sqrt{-g}\left(\frac{1}{16\pi}(R-2\lambda)+\frac{1}{4}F^2\right) \quad (28)$$

Varying this action with respect to the metric tensor leads to the well-known Tangherlini metric

$$ds^2 = -\Delta(r)dt^2 + \Delta(r)^{-1}dr^2 + r^2 d\Omega^2_{D-2} \quad (29)$$

Where

$$\Delta(r) = 1 - \frac{2m}{r^{(D-2)}} + \frac{Q^2}{r^{2(D-2)}} - \frac{r^2}{R^2} \quad (30)$$

The cosmological constant is a function of the de Sitter radius R and the dimension D. They are connected via the following relation

$$R = \sqrt{\frac{D(D-1)}{2\Lambda}} \quad (31)$$

To obtain the event horizon is not an easy task we should solve $\Delta(r) = 0$ in non-commutative backgrounds. We must invert equation this equation to get

$$r^{2D} + \alpha r^{2(D-1)} - \beta r^D + \gamma r^{D-2} - \delta r + \epsilon = 0 \quad (32)$$

The parameters appearing in this equation read as follows

$$\begin{aligned}
\alpha &= \left(\frac{\sum_{i\neq\xi}^{D-1} p_i^2\theta^2}{4} - R^2\right)\\
\beta &= 2mR^2\\
\gamma &= \frac{mR^2(2-D)}{4}\sum_{i\neq\xi}^{D-1} p_i^2\theta^2\\
\delta &= R^2Q^2\\
\epsilon &= \frac{Q^2R^2(2-D)}{4}\sum_{i\neq\xi}^{D-1} p_i^2\theta^2
\end{aligned}$$

In fact, it is not easy to find an analytical solution. It should be interesting to perform a numerical calculation to find some possible solutions for such a problem.

4. Thermodynamics of 5-dimensional Non-commutative Schwarschild Black Holes

In this section, we study the thermodynamics of non-commutative black hole solutions in arbitrary dimensions. A special emphasis is put on five dimensional solutions. In particular, we compute the Hawking temperature and the entropy functions. For simplicity reasons, we consider the simplest case corresponding to Schwarzschild black holes. More general study will be addressed elsewhere.

Instead of following an analytic description, we numerically discuss such thermodynamic proprieties. To derive that, we use the usual relations. In particular, the Hawking temperature can be obtained from

$$T_H = \frac{1}{4\pi}\frac{dg_{00}}{dr}|_{\hat{r}_{h+}} \quad (33)$$

In $D = 5$ dimensions, for instance, the expression for Hawking temperature is

$$T_H = \frac{\sqrt{2}}{\pi r_H\left(1+\sqrt{1-\frac{\theta^2\sum_{i\neq\xi}p_i^2}{4r_H^2}}\right)^{3/2}} \quad (34)$$

Choosing different values of the θ parameter, with the condition

$$\theta^2\sum_{i\neq\xi}p_i^2 = 4r_H^2 \quad (35)$$

numerical calculations show that the temperature has a nice behavior in terms of the horizon radius. The result is plotted in Fig. 1.

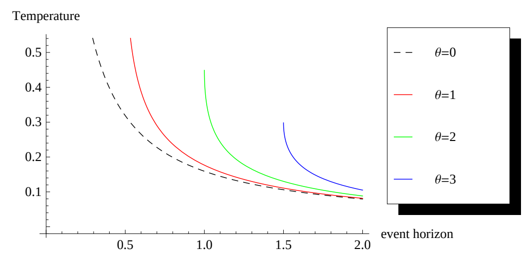

FIG. 1: The behavior of hawking temperature T_H in terms of θ and r_h.

In what follows, we consider a particular situation given by $\sum_{i\neq\xi}p_i^2 = 1$. In this way, the figure 1 shows that the maximum of the temperatures of each value of θ start at $r_h \sim \frac{\theta}{2}$. Numerically, we can easily get the following values of the maximal temperature in five dimensions. They are listed in the following table

θ ***values***	0	1	2	3	4	5	6
maximum of the temperature	∞	$\frac{2\sqrt{2}}{\pi}$	$\frac{\sqrt{2}}{\pi}$	$\frac{2\sqrt{2}}{3\pi}$	$\frac{1}{\sqrt{2}\pi}$	$\frac{2\sqrt{2}}{5\pi}$	$\frac{\sqrt{2}}{3\pi}$

Based on these values, we plot the maximal temperature in Fig. 2.

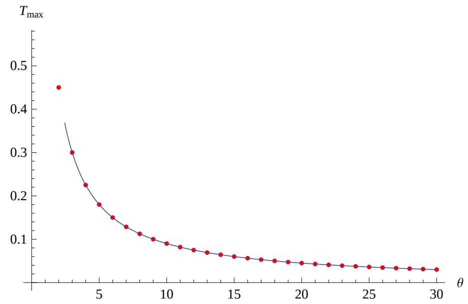

FIG. 2: The maximal temperature in terms of θ.

It is easy to see that

$$T_{max} \simeq \frac{T_0}{\theta} \tag{36}$$

Where, T_0 is the maximum of the temperature corresponding to $\theta = 1$. This is a very nice expression having a possible interpretation in string theory framework where the non-commutative geometry obtained from the implementation of NS-NS B-field [13]. It gives a linear relation between the the maximum temperature and the NS-NS B-field. Up to the above condition, the relation is given by

$$T_{max} \sim B \tag{37}$$

In string theory compactification, this indicates that the maximal temperature can be controlled by closed string backgrounds. This observation deserves a deeper study which will be considered in a future work.

Using similar calculation, the area of the horizon reads as

$$A = \frac{1}{4}\pi^2 r_H{}^3 \left(2 + \sqrt{4 - \frac{\theta^2 \sum_{i\neq\xi} p_i^2}{r_H^2}}\right)^{3/2} \tag{38}$$

Using Bekenstein-Hawking formula, we can derive the entropy, which reads as

$$S = \frac{A}{4G_5} = \frac{\pi^2 r_H{}^3}{4\sqrt{2}G_5} \left(1 + \sqrt{1 - \frac{\theta^2 \sum_{i\neq\xi} p_i^2}{4r_H^2}}\right)^{3/2} \tag{39}$$

Where, G_5 is the 5-dimensional Newton constant. It is easy to see that in the limit $\theta = 0$, we recover the usual formula of the entropy in commutative case.

As mentioned before, we calculate numerically the entropy in terms of θ. The result is plotted in Fig. 3.

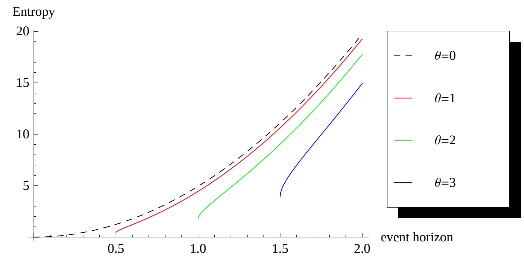

FIG. 3: The behavior of the entropy in terms of r_h for various values of θ.

A close inspection of our numerical calculation reveals that the minimum of the entropy corresponds to the following horizon radius: $r_h = \frac{\theta}{2}$ More precisely, the minimum of the entropy follows a quite similar rule. In particular, we find the results summarized in the following table.

θ ***values***	0	1	1.5	2	3	4	6
minimum of entropy	0	$\frac{\pi^2}{16\sqrt{2}}$	$\frac{9\pi^2}{64\sqrt{2}}$	$\frac{\pi^2}{4\sqrt{2}}$	$\frac{9\pi^2}{16\sqrt{2}}$	$\frac{\pi^2}{\sqrt{2}}$	$\frac{9\pi^2}{4\sqrt{2}}$

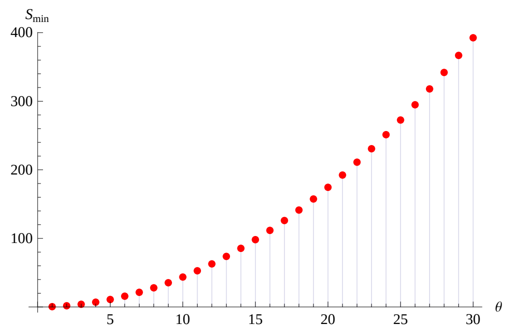

FIG. 4: The minimal entropy in terms of θ.

It is easy to see that Eqn. (39) can be reduced to

$$S_{min} = S_0 \theta^3 \tag{40}$$

Where, S_0 is the minimum of the entropy in the case $\theta = 1$. Using the minimal entropy expression and the above the maximal temperature, we find

$$S_{min} \sim \frac{1}{T_{max}^3} \tag{41}$$

Recent results in string theory have revealed a remarkable and a nice realization of the extension of the Bekenstein-Hawking results to black holes in an anti-de Sitter space. Black holes in such a background can be thermodynamically stable. It should be interesting to address this issue in future work.

5. Discussions

In this work, we have discussed the effects of non commutativity on black hole solutions in arbitrary dimensions. In particular, we have explicitly computed the horizon geometry of various black holes solutions involving extra physical constants including the cosmological constant. Then, we have dealt with some thermodynamical properties of such black hole solutions and performed numerical description of the Hawking temperature and entropy functions.

It is recalled that the parameter θ has a dimension of $(length)^2$. Choosing specifically a particular value for θ like

$$\theta = n\ell_p^2, \quad n \in \mathbb{N} \tag{42}$$

the maximum value of the temperature and the minimum entropy can
be quantified. This observation drives one to think about many open questions.

In connection with the attractor black holes in higher dimensional theories, an interesting open question concerns the contact with the attractor equations for the black object solutions. Based on this observation, Eqn. (40) could be interpreted as an attractor equation. In particular, it should be interesting to explore the non commutativity parameter in the discussion of stringy moduli stabilization. The study would be interesting for a higher dimensional model dealing with black objects where the non commutativity parameter could play the same role as the size moduli controlling the Kahler moduli space. This remark deserves deeper study. We hope these questions will be addressed in future work.

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