The Study of Bonnor Spacetime Via Gravitoelectromagnetism Approach

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The exact solutions of the Einstein field equations for the Bonnor spacetime via the gravitoelectromagnetism formalism are studied when the source of gravitational field is a perfect fluid.

1. Introduction

It is well known [1], the theory of general relativity discussed the motion of Mercury perihelion in terms of a relativistic gravitoelectric field correction to the Newtonian gravitational potential of the Sun and also it contains a gravitomagnetic field due to proper rotation of the Sun. This field influences planetary orbits [2]. The theory of gravitoelectromagnetism[2] that assumes a perfect isomorphism between gravitation and electromagnetism has been established by Heaviside [5] and Jefimenko [6]. The gravitoelectromagnetism has been discussed by a number of authors [7]. In the same way that a magnetic field is created when a charged object rotates, a gravitomagnetic field is created when a massive body rotates but this effect is too small. To detect it, it is necessary to examine a very massive object or build an instrument that is very sensitive. The gravitomagnetic field of the Earth can be measured by studying the motion of satellites LAGEOS[3] [8]. The LAGEOS measured the frame-dragging of the Earth to be 99% of the value predicted by general relativity. The theory of general relativity predicts that the rotating bodies drag spacetime around themselves in a phenomenon referred to as frame-dragging. The rotational frame-dragging effect was first derived from the theory of general relativity in 1918 by Josef Lense and Hans Thirring and is also known as the Lense-Thirring effect [2]. In 2004, Gravity Probe B [9] launched by Stanford physicists to measure the gravitomagnetism on a gyroscope in outer space with much greater precision. The data analysis of the Gravity Probe B mission is still ongoing [10]. At this time, the measurement of gravitomagnetism via superconducting gyroscopes in a satellite about the Earth is one of the aims of NASA. In this work, we rewrite the Einstein field equations in terms of gravitoelectromagnetism fields in threading formalism and then we solve these equations for the Bonnor metric.

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[2] The gravitoelectric and gravitomagnetic fields are so-called the gravitoelectromagnetism fields [3,4].

[3] LAGEOS (Laser Geodynamics Satellites) are a series of satellites designed to provide an orbiting laser ranging benchmark for geodynamical studies of the Earth.

2. Time Dependent Quasi-Maxwell Equations

The slicing and threading points of view were introduced respectively by Misner, Thorne and Wheeler [11] in 1973 and, Landau and Lifshitz [12] in 1975. Both points of view can be traced back when the Landau and Lifshitz [13] in 1941 introduced the threading point of view for splitting of the spacetime metric. After them, Lichnerowicz [14] introduced the beginnings of the slicing point of view. In 1956, Zel'manov [15] discussed the splitting of Einstein field equations in the general case. The slicing point of view is commonly referred as 3+1 or ADM formalism and also term 1+3 formalism has been suggested for the threading point of view. For more details about these formalisms, see reference [16]. In threading point of view, splitting of spacetime is introduced by a family of timelike congruences with unit tangent vector field, may be interpreted as the world-lines of a family of observers, and it defines a local time direction plus a local space through its orthogonal subspace in the tangent space. Let[4] $(M, \mathrm{g}_{\alpha\beta})$ be a 4-dim manifold of a stationary spacetime. We now can construct a 3-dim orbit manifold as $\bar{M} = \frac{M}{G}$ with projected metric tensor γ_{ij} by the smooth

[4] The Greek indices run from 0 to 3 while the Latin indices take the values 1 to 3.

map $\zeta : M \to \bar{M}$ where $\zeta(p)$ denotes the orbit of the timelike Killing vector $\frac{\partial}{\partial t}$ at the point $p \in M$ and G is 1-dim group of transformations generated by the timelike Killing vector of the spacetime under consideration [16,17]. This splitting of the spacetime leads to the following distance element [12,17]:

$$ds^2 = \mathrm{g}_{\alpha\beta} dx^\alpha dx^\beta = h(dt - \mathrm{g}_i dx^i)^2 - \gamma_{ij} dx^i dx^j \quad (1)$$

Where, x^i are local coordinates and $\gamma_{ij} = -\mathrm{g}_{ij} + h\mathrm{g}_i\mathrm{g}_j$ in which $\mathrm{g}_i = -\frac{\mathrm{g}_{0i}}{h}$ and $h = \mathrm{g}_{00}$. It is interesting to rewrite the Einstein field equations in terms of gravitoelectromagnetism fields in γ-space[5] with time dependent metric γ_{ij}. Hence, the field equations can be written as the time dependent quasi-Maxwell (abbreviated TQM) equations[6], [15,19]:

$$^*\nabla \cdot {}^*\mathbf{E} = {}^*\mathrm{E}^2 + \frac{1}{2}{}^*\mathrm{B}^2 - \frac{^*\partial \mathrm{D}}{\partial t} - d - \frac{1}{2}(\zeta + \mathrm{U}) \quad (2)$$

$$^*\nabla \times {}^*\mathbf{B} = 2({}^*\mathbf{S} + {}^*\mathbf{M} - \mathbf{j}_m) \quad (3)$$

$$\begin{aligned} ^*\mathrm{K}_{ij} &= -{}^*\nabla_{(i}{}^*\mathrm{E}_{j)} + {}^*\mathrm{E}_i{}^*\mathrm{E}_j + \frac{1}{2}({}^*\mathrm{B}_i{}^*\mathrm{B}_j - \gamma_{ij}{}^*\mathrm{B}^2) \\ &+ 2\mathrm{D}_{ik}\mathrm{D}^k_j - \mathrm{D}\mathrm{D}_{ij} + \sqrt{\gamma}\,\varepsilon_{nk(i}\mathrm{D}^n_{j)}{}^*\mathrm{B}^k \\ &- \frac{^*\partial \mathrm{D}_{ij}}{\partial t} + \mathrm{U}_{ij} + \frac{1}{2}\gamma_{ij}(\zeta - \mathrm{U}) \end{aligned} \quad (4)$$

Where, $\zeta = \frac{\mathrm{T}_{00}}{h}$ is the density of the moving substance, $\mathrm{j}^i_m = \frac{\mathrm{T}^i_0}{\sqrt{h}}$ is the momentum density, $\mathrm{U}_{ij} = \mathrm{T}_{ij}$ is 3-dimensional kinematic stress tensors and $\mathrm{U} = \mathrm{U}^i_i$, whereas $\mathrm{T}_{\mu\nu}$ is the energy-momentum tensor. Also, deformation rates of the reference frame with respect to the observer are represented by the following tensors

$$\begin{aligned} \mathrm{D}_{ij} &= \frac{1}{2}\frac{^*\partial\gamma_{ij}}{\partial t}, \ \mathrm{D}^{ij} = -\frac{1}{2}\frac{^*\partial\gamma^{ij}}{\partial t} \\ \mathrm{D} &= \gamma^{ij}\mathrm{D}_{ij} = \frac{^*\partial \ln\sqrt{\gamma}}{\partial t} \end{aligned} \quad (5)$$

Where, $\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{h}}\frac{\partial}{\partial t}$ and $\gamma = \det(\gamma_{ij})$. In TQM equations, $d = \mathrm{D}_{ij}\mathrm{D}^{ij}$ and time dependent gravitoelectromagnetism fields are defined in terms of gravoelectric potential $\phi = \ln\sqrt{h}$ and gravomagnetic vector potential $\mathrm{g} = (\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3)$ as follows[7]

$$^*\mathbf{E} = -{}^*\nabla\phi - \frac{\partial \mathrm{g}}{\partial t}\ ; \ {}^*\mathrm{E}_i = -\phi_{*i} - \frac{\partial \mathrm{g}_i}{\partial t} \quad (6)$$

$$\frac{^*\mathbf{B}}{\sqrt{h}} = {}^*\nabla \times \mathrm{g}\ ; \ \frac{^*\mathrm{B}^i}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\mathrm{g}_{[k*j]} \quad (7)$$

In Eqn. (4), $^*\mathrm{K}_{ij}$ is 3-dimensional starry Ricci tensor constructed from 3-dim starry Christoffel symbols as $^*\mathrm{K}_{ij} = {}^*\lambda^k_{ij*k} - {}^*\lambda^k_{ik*j} + {}^*\lambda^n_{ij}{}^*\lambda^k_{kn} - {}^*\lambda^n_{ik}{}^*\lambda^k_{nj}$ where $^*\lambda^i_{jk} = \frac{1}{2}\gamma^{il}(\gamma_{jl*k} + \gamma_{kl*j} - \gamma_{jk*l})$ and also the starry covariant derivatives of an arbitrary 3-vector and a tensor are given by $^*\nabla_j\mathrm{A}_i = \mathrm{A}_{i*j} - {}^*\lambda^k_{ij}\mathrm{A}_k$ and $^*\nabla_k\mathrm{T}^{ij} = \mathrm{T}^{ij}_{*k} + {}^*\lambda^i_{nk}\mathrm{T}^{jn} + {}^*\lambda^j_{nk}\mathrm{T}^{in}$. Finally, the vectors $^*\mathbf{S} = {}^*\mathbf{E} \times {}^*\mathbf{B}$ and $^*\mathbf{M}$ have components as $^*\mathrm{S}^i = \frac{\varepsilon^{ijk}}{\sqrt{\gamma}}{}^*\mathrm{E}_j{}^*\mathrm{B}_k$ and $^*\mathrm{M}^i = -{}^*\nabla_j\mathrm{D}^{ij} + {}^*\partial^i\mathrm{D}$. For more details about the application of gravitoelectromagnetism fields, see references [19,20].

3. Exact Solutions of the Bonnor Metric

We assume that the spacetime metric is of the Bonnor form and in the cylindrical coordinates it has the following line element, [21].

$$ds^2 = r^{2m^2}\mathrm{G}^2(dt^2 - dr^2) - r^2\mathrm{G}^2 d\phi^2 - \mathrm{G}^{-2}dz^2 \quad (8)$$

Where, G is an unknown function of r and m is a constant. Firstly, it is easy to check that all components of gravitoelectromagnetism fields are vanish, except

$$^*\mathrm{E}_{g1} = -\frac{\mathrm{G}'}{\mathrm{G}} - \frac{m^2}{r} \quad (9)$$

Where, the over head prime indicate the partial differentiation with respect to r. For later use, we will need the following relation

$$\begin{aligned} ^*\nabla \cdot {}^*\mathbf{E} &= r^{-2m^2}\mathrm{G}^{-2}\left(-\frac{\mathrm{G}''}{\mathrm{G}} + 2\left(\frac{\mathrm{G}'}{\mathrm{G}}\right)^2 \right. \\ &\left. + \frac{2m^2 - 1}{r}\frac{\mathrm{G}'}{\mathrm{G}} + \frac{m^4}{r^2}\right) \end{aligned} \quad (10)$$

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[5] The quotient space obtained by quotienting spacetime by the action of the stationary isometry and it represents the collection of the orbits of the Killing vectors, [18].

[6] The symbols () and [] represent the commutation and anticommutation over indices, gravitational units with c=G=1 are used and the 3-dim Levi-Civita tensor ε_{ijk} is antisymmetric under interchange of any pair of indices such that $\varepsilon_{123} = \varepsilon^{123} = 1$ [12]. Also, we note that $^*\mathrm{E}^2_g = \gamma^{ij}\,{}^*\mathrm{E}_{gi}{}^*\mathrm{E}_{gj}$.

[7] Note that divergence and curl of an arbitrary vector in γ-space are defined by $^*\nabla \cdot \mathbf{A} = \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}\,\mathrm{A}^i)_{*i}$ and $(^*\nabla \times \mathbf{A})^i \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\mathrm{A}_{[k*j]}$ while $_{*i} = {}^*\partial_i = \partial_i + \mathrm{g}_i\frac{\partial}{\partial t}$.

Also, after some work, we can obtain

$$\begin{aligned}
{}^{*}\lambda^{1}_{11} &= \frac{\mathrm{G}'}{\mathrm{G}} + \frac{m^2}{r} \\
{}^{*}\lambda^{1}_{22} &= -r^{-2m^2+1}(1 + r\frac{\mathrm{G}'}{\mathrm{G}}) \\
{}^{*}\lambda^{1}_{33} &= r^{-2m^2}\frac{\mathrm{G}'}{\mathrm{G}^5} \\
{}^{*}\lambda^{2}_{12} &= \frac{\mathrm{G}'}{\mathrm{G}} + \frac{1}{r} \\
{}^{*}\lambda^{3}_{13} &= -\frac{\mathrm{G}'}{\mathrm{G}}
\end{aligned} \tag{11}$$

and other symbols not listed above are zero. Applying these symbols again, we can derive the following expression

$$ {}^{*}\mathrm{K}_{ij} = \begin{cases} -2(\frac{\mathrm{G}'}{\mathrm{G}})^2 - \frac{1}{r}\frac{\mathrm{G}'}{\mathrm{G}} + \frac{m^2}{r^2} & i,j=1, \\ r^{-2m^2+2}(-\frac{\mathrm{G}''}{\mathrm{G}} + 2(\frac{\mathrm{G}'}{\mathrm{G}})^2 \\ \quad +\frac{m^2}{r}\frac{\mathrm{G}'}{\mathrm{G}} + \frac{m^2}{r^2}) & i,j=2, \\ r^{-2m^2}\mathrm{G}^{-4}(\frac{\mathrm{G}''}{\mathrm{G}} - 2(\frac{\mathrm{G}'}{\mathrm{G}})^2 \\ \quad +\frac{1-m^2}{r}\frac{\mathrm{G}'}{\mathrm{G}}) & i,j=3, \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

In this analysis, we assume that the matter content is a perfect fluid, i.e.,

$$\mathrm{T}_{\mu\nu} = (\rho + p)u_\mu u_\nu - p\mathrm{g}_{\mu\nu} \tag{13}$$

Where, ρ, p and u_μ are respectively the matter density, isotropic pressure and four velocity vector of the matter distribution with co-moving coordinates as $u_\mu = (1,0,0,0)$. At this stage, it is clear that the equation (3) is trivial. If we substitute Eqns. (10), (12) and (13) into the field equations (2) and (4), which then leads to the following equations, respectively.

$$\frac{\mathrm{G}''}{\mathrm{G}} - (\frac{\mathrm{G}'}{\mathrm{G}})^2 + \frac{1}{r}\frac{\mathrm{G}'}{\mathrm{G}} - \frac{\rho + p(1 + 2r^{2m^2}\mathrm{G}^2)}{2} = 0 \tag{14}$$

$$\frac{\mathrm{G}''}{\mathrm{G}} + (\frac{\mathrm{G}'}{\mathrm{G}})^2 + \frac{1}{r}\frac{\mathrm{G}'}{\mathrm{G}} - \frac{2m^2}{r^2} + \frac{\rho + p(1 - 2r^{2m^2}\mathrm{G}^2)}{2} = 0 \tag{15}$$

$$\frac{\mathrm{G}''}{\mathrm{G}} - (\frac{\mathrm{G}'}{\mathrm{G}})^2 + \frac{1}{r}\frac{\mathrm{G}'}{\mathrm{G}} + \frac{\rho + p(1 - 2r^{2m^2}\mathrm{G}^2)}{2} = 0 \tag{16}$$

$$\frac{\mathrm{G}''}{\mathrm{G}} - (\frac{\mathrm{G}'}{\mathrm{G}})^2 + \frac{1}{r}\frac{\mathrm{G}'}{\mathrm{G}} - \frac{\rho + p(1 - 2r^{2m^2}\mathrm{G}^2)}{2} = 0 \tag{17}$$

Where, the quantities ρ and p depend on r. A comparison of Eqns. (16) and (17) yields

$$\frac{\mathrm{G}''}{\mathrm{G}} - (\frac{\mathrm{G}'}{\mathrm{G}})^2 + \frac{1}{r}\frac{\mathrm{G}'}{\mathrm{G}} = 0 \tag{18}$$

The solution of this equation is

$$\mathrm{G} = ar^n \tag{19}$$

Where, a and n are constants. Also, from Eqns. (14), (16) and (17), we can conclude

$$\rho = p = 0 \tag{20}$$

which is equivalent to the vacuum solution. Finally, if we substitute Eqns. (19), (20) into Eqn. (15), we obtain the following result

$$m = \pm n \tag{21}$$

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