

A Signal-Processing Interpretation of Quantum Mechanics

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We present a new model to explain the behavior of transmitted quantum particles, by analogy with a wireless communication system. The particle's complex wavefunction is interpreted as the amplitude and phase of a modulated carrier wave. Particle transmission events are modeled as the outcome of a process of signal accumulation that occurs in an extra (non-spacetime) dimension. The standard probability density interpretation of the squared amplitude of the wavefunction is a derivable consequence of the model. The so-called "collapse of the wave packet" also has a simple interpretation within the model's framework. We simulate the model for a 2-slit diffraction experiment, and indicate possible deviations of the model's predictions from conventional quantum mechanics.

1. Introduction

The behavior of nonrelativistic quantum particles is conventionally described by the Schrödinger wave equation [1]

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x},t) + V(\vec{x},t)\psi = -i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) \quad (1)$$

Where, \hbar is the Planck's constant, m is the particle's mass, \vec{x}, t denote spatial position and time, respectively, $V(\vec{x}, t)$ is a potential function that may depend on space and time, and $\psi(\vec{x}, t)$ is the wavefunction for the quantum particle. The wavefunction ψ is strikingly different from classical waves (such as electromagnetic waves) in at least two respects. First, ψ is fundamentally complex. (It is true that electromagnetic wave equations in some situations can also be expressed in complex format, but these are mathematical re-expressions of the fundamental real equations.) Second, the field magnitude $\psi(\vec{x}, t)$ has no direct physical interpretation. Rather, according to the empirical *Born rule*, the squared field magnitude equation $|\psi(\vec{x}, t)|^2$ gives the probability density for discrete particle detection events

Since the beginning of quantum mechanics, physicists have struggled to understand the physical mechanism that gives rise to these two properties. However, in wireless digital communications one does encounter systems that exhibit both characteristics. This paper shows how quantum pro-

perties could be due to an underlying structure similar to a conventional wireless communication system.

The paper is organized as follows. In Sec. 2 we describe a communication system that possesses "quantum" characteristics. In Sec. 3 we modify this system to produce a model of quantum particle detection. In Sec. 3 we discuss physical consequences of the model and in Sec. 6 we summarize our conclusions.

2. Wireless System Model

Consider a mobile receiver moving randomly within a region in which a modulated carrier wave is broadcast, as shown in Fig. 1. The carrier wave is modulated both in amplitude and phase. In order to detect the broadcasted signal, the receiver accumulates its received signal until a detection threshold is reached. In our model, the wireless signal has the following characteristics:

- The carrier frequency is ω , so that the signal has the general mathematical form $A(\vec{x}, t)\cos(\omega t + \phi(\vec{x}, t))$ and is conventionally represented by its "complex amplitude" $A(\vec{x}, t)e^{i\phi(\vec{x}, t)}$.
- The transmitted signal (at the transmitter) has constant complex amplitude over time intervals of length δ , where $\delta \gg 2\pi/\omega$ (δ is the "chip width" [2]). The probability distribution of complex amplitudes is Gaussian, so that real and imaginary parts are independent, identically distributed (i.i.d) standard normal random variables with mean 0 and variance 1.

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- The ratio of field amplitude to transmitted signal amplitude (denoted by $\psi(\vec{x})$) depends on the field location \vec{x} , but is independent of time (corresponding to a static propagation environment). For mathematical simplicity, we assume that we may partition the region into K subregions of equal area such that $\psi(\vec{x})$ is constant on each region (see Fig. 1). That is, $\psi(\vec{x}) = \psi_k$ for all x in region $k, k = 1, \dots, K$. By taking finer and finer partitions, we may recover the continuum limit.

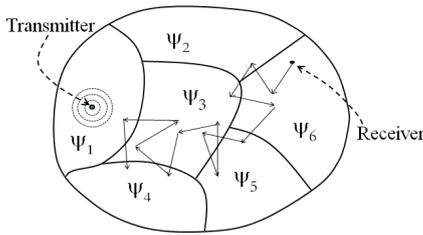


FIG. 1: Wireless system model.

The receiver has the following characteristics:

- The receiver consists of an oscillating circuit with natural frequency ω , which is driven by the signal field at the receiver's current location.
- The receiver's location is denoted by $\vec{r}(t)$, and the receiver moves slowly enough so that $\psi(\vec{r}(t))$ can be considered to be constant over time intervals of length $M\delta$, where M is an integer $\gg 1$.
- The receiver moves in such a way that its position uniformly samples the entire region of interest (for instance, by random walk).
- Our mathematical proof (see Appendix A) requires that the receiver's fields over the time intervals $(m_1, m_1 + 1)M\delta$ and $(m_2, m_2 + 1)M\delta$ are statistically independent whenever $m_1 \neq m_2$. Strictly speaking, a receiver moving under random walk will not satisfy this condition. Instead, the receiver would have to make uniformly-distributed random jumps at times $M\delta, 2M\delta, \dots$. A rigorous treatment with random-walk motion would require a more careful analysis.
- The receiver detects the signal when the receiver's amplitude exceeds a fixed threshold Θ' , where $\Theta' \gg \frac{\delta}{2\omega} M \max|\psi|$.

Let the random variable \vec{r}^* represent the position of the receiver at the instant of detection. In Appendix A we prove that under the above assumptions we obtain the following probability distribution:

$$\Pr[\vec{r}^* \in \text{Region } k] \propto |\psi_k|^2, \quad k = 1, \dots, K \quad (2)$$

which is a discretized version of the Born rule. That is, the probability density for detection at a given location is proportional to the squared magnitude of the wavefunction at that location.

3. Single Quantum Detection Event Model

In this section, we modify the previous model to obtain a model for quantum single-particle transmission. The model is discretized for simplicity, but it is straightforward to see how the model can be taken to a continuous limit.

We emphasize that the probability distribution in the previous model arose from the outcome of a process that involves sampling the entire region of potential detection before the actual detection was made. This representative sampling was necessary in order for the field strengths to translate into relative probabilities. We want similar characteristics for the quantum process.

In the wireless scenario presented above the process variable was time. This was appropriate because we were only concerned about the spatial position of the receiver at the moment of detection. However, in quantum mechanics, we are concerned about the location of detection events within space-time. It is impossible to have a process that unfolds in time, that also samples all space-time locations before determining the detection location. For this reason, it is necessary to introduce a new process variable so that the process of signal accumulation takes place in a non-observable dimension, which we will call the a -dimension.

We also postulate a carrier wave that oscillates as a function of a (*not* as a function of time) having the mathematical form $\sin \omega a$. The frequency ω is unknown, and does not correspond to any measurable quantity in space-time. The signal has the following characteristics:

- The signal has constant complex amplitude over a -intervals of length δ , where $\delta \gg 2\pi/\omega$. The distribution of complex amplitudes is mean-zero Gaussian, with i.i.d. standard normal real and imaginary parts.
- The signal is multiplied by a complex field amplitude $\psi(\vec{r}, t)$, which is independent of a .

For mathematical simplicity, we assume that the amplitude takes one from a finite set of complex values $\{\psi_1, \dots, \psi_K\}$ and that within the space-time confines of the detector. The sets $\{\vec{r}, t | \psi(\vec{r}, t) = \psi_k\}$ ($k = 1, \dots, K$) all have equal 4-volume.

We suppose that the detection environment of the particle is fixed. In this basic model, the detection environment consists of a set of *detection locations*, which are space-time points where a particle detection could possibly occur. In the context of a scattering experiment, the detection locations might correspond to atoms in a detection screen.

The previous scenario had a physical receiver which moved within the state space of possible detection locations. Quantum detection (say of a particle on a screen) does not appear to have any corresponding receiver. We therefore introduce the conceptual notion of a *detectron*, which plays the same role as the receiver in our previous model. Signal accumulation takes place as the detectron moves around and uniformly samples the set of all potential detection locations. This motion takes place in the a -dimension; for fixed a , the detectron's space-time location is fixed. In this respect the detectron should not be considered as a moving material particle, but rather as a kind of placeholder for possible detection sites.

The detectron has the following characteristics:

- Associated with the detectron is an oscillator (which varies sinusoidally with a) with natural frequency ω , which is driven by the signal field at the detectron's current space-time location.
- The detectron moves in space-time (as a function of a) slowly enough so that its field amplitude does not change significantly over a -intervals of length $M\delta$, where M is an integer $\gg 1$.
- The detectron moves in such a way that it uniformly samples the space-time extent of the detector.
- A particle detection occurs when the detectron's oscillator's amplitude exceeds a fixed threshold $\Theta' \gg \frac{\delta}{2\omega} M \max|\psi|$.

We can apply this model to the two-slit diffraction experiment shown in Fig. 2. The detectron's location (as a function of a) uniformly samples the space-time locations corresponding to the detection screen. The complex field amplitude $\psi(\vec{r}, t)$

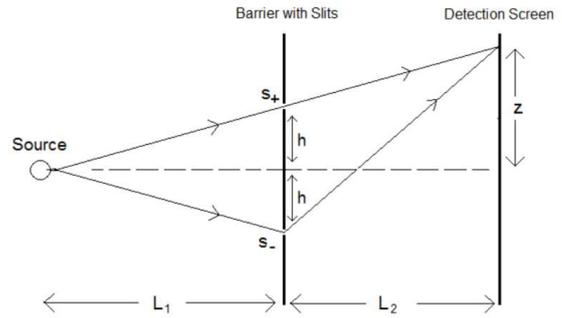


FIG. 2: Notation for quantum two-slit experiment.

corresponds to the conventional Schrödinger wavefunction at the screen, which in the ray approximation is given by

$$\psi(0, L, z, t) \propto d_1^{-1} e^{i(k'd_1 - \omega't)} + d_2^{-1} e^{i(k'd_2 - \omega't)} \quad (3)$$

Where, k' , ω' are the (observable) wave number and frequency, and $d_1 = (L_1^2 + h^2)^{1/2} + (L_2^2 + (z - h)^2)^{1/2}$, $d_2 = (L_1^2 + h^2)^{1/2} + (L_2^2 + (z + h)^2)^{1/2}$. We simulated this system using MATLAB, with the physical lengths $h = 5, L_1 = 10^4, L_2 = 10^6$, and $z_{max} = 10^6$ (all measured in observable wavelengths). We considered a single time slice $t = 0$, and restricted to the $x = 0$ portion of the screen. The z locations were discretized into 100 bins, and the detectron jumped uniformly randomly from bin to bin every $M = 400$ iteration steps. At iteration n , the signal was incremented by $\nu_n \cdot \psi(0, L, z, 0)$, where the $\{\nu_n\}$ are i.i.d. complex random variables with standard normal real and imaginary parts. Each time the detection threshold $\Theta = 500$ was reached, a detection was logged and the simulation was restarted. Altogether 100,000 detections were logged. Fig. 3 shows the detection probability distribution obtained in the simulation. The agreement is very close with the theoretical result $|\psi(z)|^2$, with ψ given by (3).

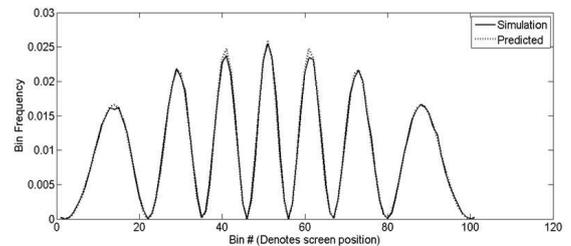


FIG. 3: Simulation and theory for double-slit experiment.

4. Physical Implications of the Model

Our model suggests that the usual formula for a quantum wavefunction is a statistical approximation, and small deviations from the probabilities predicted by the wave equation should be expected. In particular, detection rates near theoretical wavefunction nulls should be higher than the conventional quantum predictions because of random fluctuations in the accumulation process. This effect was indeed observed in the simulation (see Fig. 3). The simulation also showed that distribution peaks are slightly lower than theoretical values. Unfortunately we cannot make specific numerical predictions for the magnitude of these effects, because they depend on aspects of the process that cannot be measured directly (this is analogous to the historical situation with Boltzmann's constant, which was first determined indirectly by Perrin [3] via the equilibrium distribution of particles in colloidal suspension).

In our model, the wave function is seen as an "actual" field, and not merely a representation of the observer's partial knowledge. On the other hand, the field is not directly observable via physical events in space-time. The apparent "collapse" of the wave packet [1] is due to the fact that the observable universe is a single "*a*-slice" of the entire process.

Quantum mechanics is well-known for exhibiting nonlocal phenomena: in other words, different quantum events may be correlated even though they are separated in space and time in such a way that (relativistically speaking) no information can pass between them. The EPR effect and Bell's inequality are prominent examples of quantum nonlocality [1]. This paradoxical behavior is consistent with our model, because the process of accumulation is nonlocal: the detectron samples *all* potential detection locations during the process of determining the actual detection location.

Admittedly our model is incomplete because it does not explain the formation of the signal, nor the existence of the detectron. Also, it is restricted to single-particle detection in a fixed environment. We shall address these limitations in our future research.

5. Comparison With Other Interpretations of Quantum Mechanics

Several physicists have proposed alternative interpretations of quantum mechanics. In this section we briefly compare our interpretation with the most prominent of these other interpretations.

Everett's "Many-worlds" interpretation [4] requires that space-times multiply exponentially as a function of time, and "all possible" universes exist in parallel. Our model, which embeds our space-time universe within one additional dimension, possesses a vastly smaller and simpler state space.

Bohm's quantum mechanics [5] posits that particles such as electrons are able to track along with pilot waves. This appears to imply that these particles have some sort of inner structure. In our model particles are not "objects" at all, so no such complications appear.

Cramer's transactional quantum mechanics [6] interprets ψ^* as a wave traveling backwards in time, but gives no explanation why $\psi\psi^*$ should be interpreted as a probability. Furthermore, transactional quantum mechanics is not very clear about the order in which "transactions" are determined. In our model, all transactions are determined "simultaneously" (at $a = a\Theta$), and a single accumulation process is used to determine all interaction events.

We also remark that none of these alternative models explains why the wavefunction is complex, nor why the squared amplitude is interpreted as a probability.

6. Conclusions

This model would represent a radically different picture of quantum particle transmission. The wavefunction is given a physical interpretation in terms of a signal field; and the usual quantum mechanical probability density derived from the wavefunction is the natural result of a thresholding process involving this field. There is no traveling particle at all, only a process of accumulation that culminates in a detection event. This process is not "causal" in the usual sense that we commonly presume that the present is determined by the past. Rather, present events are the outcome of a process that samples all times past and future. The appearance of temporal causality in everyday physics is due to correlation and not causation.

The model predicts that detection probabilities near the nulls of interference patterns should be higher than those predicted by conventional quantum mechanics, and that interference peak probabilities should be lowered. However, so far we cannot predict the size of these effects.

APPENDIX A: Derivation of Fundamental Result

With the assumptions presented in Sec. 2, the field at space position \vec{x} at time t can be expressed (with the aid of complex amplitudes) as follows:

$$A(\vec{x}, t) = \text{Re}[\psi(\vec{x}) \cdot \nu_{\lceil t/\delta \rceil} \cdot e^{i\omega t}] \quad (\text{A1})$$

Where, $\psi(\vec{x})$ is the ratio of field amplitude at \vec{x} to signal amplitude at the transmitter; $\nu_{\lceil t/\delta \rceil} \equiv \nu_n = \alpha_n + i\beta_n$, where α_n, β_n are i.i.d. standard normal random variables; and $\lceil y \rceil$ denotes the ‘‘ceiling’’ function.

We now suppose that the trajectory of the receiver is given by the function $\vec{r}(t)$. It follows that the equation for the amplitude $x(t)$ of the driven oscillator is

$$x'' + \omega^2 x = A(\vec{r}(t), t) \quad (\text{A2})$$

This equation may be expressed as the real part of the complex equation

$$z'' + \omega^2 z = \psi(\vec{r}(t)) \cdot \nu_{\lceil t/\delta \rceil} \cdot e^{i\omega t} \quad (\text{A3})$$

The solution of (A3) which satisfies $z(0) = z'(0) = 0$ is

$$z(t) = \frac{-i}{2\omega} \int_0^t \psi(\vec{r}(u)) \cdot \nu_{\lceil u/\delta \rceil} du \cdot e^{i\omega t} + \frac{i}{2\omega} \int_0^t \psi(\vec{r}(u)) \cdot \nu_{\lceil u/\delta \rceil} \cdot e^{2i\omega u} du \cdot e^{-i\omega t} \quad (\text{A4})$$

According to our assumptions, the factor $e^{2i\omega u}$ in the second integrand oscillates rapidly compared to the rest of the integrand, which causes the second integral to be negligible compared to the first. Also, the model assumptions imply that $\psi(\vec{r}(u))$ can be treated as constant over time intervals of length $M\delta$. Using the notation $\Psi_{\lceil u/(M\delta) \rceil} \equiv \psi(\vec{r}(u))$, we then have:

$$z(t) \approx \frac{-i\delta}{2\omega} \sum_{n=1}^{\lceil t/\delta \rceil} \Psi_{\lceil n/M \rceil} \cdot \nu_n \cdot e^{i\omega t} \quad (\text{A5})$$

The oscillation at time $N\delta$ has complex amplitude $(-i\delta/2\omega) \cdot S(N)$, where

$$S(N) \equiv \sum_{n=1}^N \Psi_{\lceil n/M \rceil} \cdot \nu_n \quad (\text{A6})$$

According to the assumptions of this model, each Ψ_j is one of the values $\{\psi_1, \dots, \psi_K\}$. Define

$$\Theta \equiv \frac{2\omega}{\delta} \Theta' \quad (\text{A7})$$

$$N_\Theta \equiv \min_N \{N \mid |S(N)| \geq \Theta\} \quad (\text{A8})$$

$$\kappa(m) \equiv \{k \mid \Psi_m = \psi_k\} \quad (\text{A9})$$

Our goal is to evaluate the probability distribution of $\kappa(\dots)$ corresponding to the first passing of the threshold Θ :

$$\text{Pr}[\kappa(\lceil N_\Theta/M \rceil) = k] \quad k = 1, \dots, K \quad (\text{A10})$$

In order to investigate the dependence of $\text{Pr}[\kappa(\lceil N_\Theta/M \rceil) = k]$ on ψ_k , for each fixed $m' > 0$ we will investigate the event

$$E_{m',k} \equiv [(m' = \lceil N_\Theta/M \rceil) \wedge (\kappa(m') = k)] \quad (\text{A11})$$

(here and elsewhere ‘‘ \wedge ’’ denotes logical ‘‘and’’) conditioned on fixed sequences of $(m' - 1)$ initial ψ 's, corresponding to the $K^{m'-1}$ events

$$F_{m'}(\{k'_1, \dots, k'_{m'-1}\}) \equiv \{ \kappa(m) = k'_m, 1 \leq m < m', \}, 1 \leq k'_m \leq K \quad (\text{A12})$$

The key step in the proof shall be proving that

$$\text{Pr}[E_{m',k} | F_{m'}(\{k'_1 \dots k'_{m'-1}\})] \approx C(m', \{k'_1 \dots k'_{m'-1}\}) |\psi_k|^2 \quad (\text{A13})$$

Where, $C(\dots)$ is independent of k . The events $\{F_{m'}(\{k'_1, \dots, k'_{m'-1}\})\}$ for fixed m' partition the sample space and $\text{Pr}[F_{m'}(\{k'_1, \dots, k'_{m'-1}\})] = K^{1-m'}$. Furthermore, the events $\{E_{m',k}\}_{m'=1,2,\dots}$ partition the event $\{\kappa(\lceil N_\Theta/M \rceil) = k\}$, so from (A13) we obtain

$$\text{Pr}[\kappa(\lceil N_\Theta/M \rceil) = k] \sum_{m'} \sum_{k'_1 \dots k'_{m'}} \text{Pr}[E_{m',k} | F_{m'}(\{k'_1 \dots k'_{m'-1}\})] \text{Pr}[F_{m'}(\{k'_1 \dots k'_{m'-1}\})] \quad (\text{A14})$$

$$\approx |\psi_k|^2 \sum_{m'} \sum_{k'_1 \dots k'_{m'}} C(m', k'_1 \dots k'_{m'-1}) \cdot K^{1-m'} \quad (\text{A15})$$

which is proportional to $|\psi_k|^2$ as claimed.

We prove (A13) as follows. We shall re-express the event $E_{m',k}$ as

$$E_{m',k} = \bigcup_{0 \leq r \leq 1} [N_\Theta \leq m'M \wedge \kappa(m') = k] \wedge [(S((m' - 1)M) = r\Theta) \wedge (\lceil N_\Theta/M \rceil > m' - 1)] \tag{A16}$$

so that

$$\begin{aligned} & \Pr[E_{m',k} | F_{m'}(\{k'_1 \dots k'_{m'-1}\})] \\ &= \int_{r=0}^{r=1} \Pr[(N_\Theta \leq m'M) | (\kappa(m') = k) \wedge (S((m' - 1)M) = r\Theta)] \cdot \Pr[\kappa(m') = k] \\ & \cdot dP[(S((m' - 1)M) = r\Theta) \wedge (\lceil N_\Theta/M \rceil > m' - 1) | F_{m'}(\{k'_1 \dots k'_{m'-1}\})] \\ &= K^{-1} \int_{r=0}^{r=1} (1 - \Pr[(N_\Theta > m'M) | (\kappa(m') = k) \wedge (S((m' - 1)M) = r\Theta)]) \\ & \cdot dP[(S((m' - 1)M) = r\Theta) \wedge (\lceil N_\Theta/M \rceil > m' - 1) | F_{m'}(\{k'_1 \dots k'_{m'-1}\})] \end{aligned} \tag{A17}$$

We first evaluate the $dP[\dots]$ term in (A17) as follows. Conditioned on event $F_{m'}(\{k'_1, \dots, k'_{m'-1}\})$, we have

$$S(N) = \sum_{n=1}^N (\psi_{k'_{\lceil n/M \rceil}} \cdot \nu_n), \quad (N \leq m'M) \tag{A18}$$

Where, the $\{\nu_n\}$ have i.i.d. standard normal real and imaginary parts (we write this as $\nu_n \sim N(0, 1) + iN(0, 1)$). It follows that $S(N)$ is a random walk in the complex plane with independent (but not identically distributed) steps. We also have

$$E[|S(N)|^2] = 2 \sum_{n=1}^N \left| \psi_{k'_{\lceil n/M \rceil}} \right|^2 \tag{A19}$$

Where, $E[\dots]$ denotes expected value. Since we have assumed that $\Theta \gg \max_k |\psi_k|$, it follows that Θ is much greater than any individual term in $S(N)$, and the distribution of $\{\Theta^{-1}S(N)\}$ for all sample paths S can be approximated in probability by a standard Brownian motion $B(\tau)$ [7],

where the time variable τ is given by

$$\tau(N) \equiv E[|\Theta^{-1}S(N)|^2] \approx 2\Theta^{-2} \sum_{n=1}^N \left| \psi_{k'_{\lceil n/M \rceil}} \right|^2 \tag{A20}$$

The sample paths included in the event $\lceil N_\Theta/M \rceil \geq m'$ correspond to Brownian motion paths for which $|B(t)| < 1$ for all $t \leq \tau((m' - 1)M)$. Restricted to these paths, the distribution of $B(\tau((m' - 1)M))$ corresponds to the position probability density for a standard Brownian motion with absorbing barrier at $|z| = 1$. Let $\beta(z, T)$ be the probability density at time T of a Brownian motion with absorbing barrier at $|z| = 1$. Then $\beta(z, T)$ is the solution to the diffusion equation with boundary conditions $\beta(z, T) = 0$ for $|z| = 1$ and initial conditions $\beta(z, 0) = \delta(z)$, where $\delta(\dots)$ is the Dirac delta function [7]. The solution may be expressed as a series expansion in the Bessel functions $\{J_0(\alpha_n r)\}_{n=1,2,\dots}$ with time-dependent coefficients. The solution is radial (so we may write $\beta(z, T)$ as $\beta(r, T)$) and satisfies $\beta_r(1, T) < 0$ for all $T > 0$. (These properties can be mathematically proven, but are also intuitive consequences of the physical interpretation of $\beta(r, T)$ as an evolving temperature distribution within a disk where the boundary is held at zero temperature.) It follows that $\beta(r, T)$ can be approximated near the boundary $|z| = 1$ as

$$\beta(r, T) = (1 - r)|\beta_r(1, T)| + O[(1 - r)^2] \tag{A21}$$

and our identification of $\{S(N)/\Theta\}$ with $B(\tau(N))$ gives

$$\begin{aligned} & dP[(|S((m' - 1)M)| = r\Theta) \wedge (\lceil N_\Theta/M \rceil > m' - 1)] \\ & \approx (A(1 - r) + O(1 - r)^2)dr \end{aligned} \tag{A22}$$

Where, A is independent of r .

We next evaluate the integrand in (A17). Let $\zeta \equiv \arg[S((m' - 1)M)]$, and rotate in the complex plane by an angle $-\zeta$ to obtain

$$\begin{aligned}
 & \Pr[(N_\Theta > m'M) \mid (\kappa(m') = k) \wedge (S((m' - 1)M) = r\Theta)] \\
 &= \Pr[|S(N)| < \Theta \forall N \in \{(m' - 1)M + 1 \dots m'M\} \\
 & \quad \mid (\kappa(m') = k) \wedge (|S((m' - 1)M)| = r\Theta)] \\
 &= \Pr \left[\left| S((m' - 1)M) + \psi_k \sum_{n=(m'-1)M+1}^N \nu_n \right| \right. \\
 & \quad \left. < \Theta \forall N \in \{(m' - 1)M + 1 \dots m'M\} \right. \\
 & \quad \left. \mid (|S((m' - 1)M)| = r\Theta) \right] \\
 &= \Pr \left[\left| r\Theta + \psi_k e^{-i\zeta} \sum_{n=(m'-1)M+1}^N \nu_n \right| \right. \\
 & \quad \left. < \Theta \forall N \in \{(m' - 1)M + 1 \dots m'M\} \right] \\
 &= \Pr \left[\left| \frac{\Theta}{|\psi_k|} - x + \sum_{n=1}^N \mu_n \right| < \frac{\Theta}{|\psi_k|} \forall N \in \{1 \dots M\} \right] \tag{A23}
 \end{aligned}$$

where

$$x \equiv (1 - r)\Theta/|\psi_k| \tag{A24}$$

$$\mu_n \equiv e^{i(\arg(\psi_k) - \zeta)} \nu_{n+(m'-1)M}. \tag{A25}$$

Note that $\mu_n \sim N(0, 1) + iN(0, 1)$, so that $\text{Re } \mu_n \sim N(0, 1)$. In the limit as $|\psi_k|/\Theta \rightarrow 0$ (for fixed M), we have (see Fig. 4)

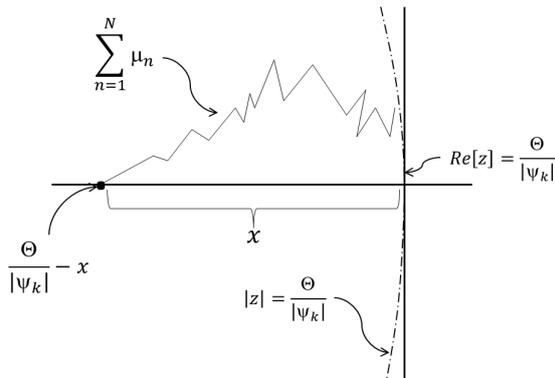


FIG. 4: Illustration of limit in (A26): $\Theta/|\psi_k| \gg 1$.

$$\begin{aligned}
 & \lim_{|\psi_k|/\Theta \rightarrow 0} \Pr \left[\left| \frac{\Theta}{|\psi_k|} - x + \sum_{n=1}^N \mu_n \right| \right. \\
 & \quad \left. < \frac{\Theta}{|\psi_k|} \forall N \in \{1 \dots M\} \right] \\
 &= \Pr \left[\sum_{n=1}^N \text{Re}(\mu_n) < x \forall N \in \{1 \dots M\} \right] \tag{A26}
 \end{aligned}$$

Where, $\mu_n \equiv \text{Re}[e^{i(\arg \psi_k - \zeta)} \nu_{n+(m'-1)M}]$ and $\mu_n \sim N(0, 1)$. By the monotone convergence theorem for integrals, for sufficiently large $\Theta/|\psi_k|$ we may replace the integrand in (Eqn. (A17)) with one minus the right-hand side of (Eqn. (A26)) to any given accuracy. Recalling the definition of x in (Eqn. (A24)), we have then from Eqn. A17 and Eqns. (A22) - (A26) that

$$\begin{aligned}
 & \Pr[E_{m',k} | F'_m(\{k'_1, \dots, k'_{m'-1}\})] \\
 & \approx K^{-1} \int_0^\infty \left(1 - \Pr \left[\sum_{n=1}^N \text{Re}(\mu_n) \right. \right. \\
 & \quad \left. \left. < x \forall N \in \{1 \dots M\} \right] \right) \\
 & \cdot A \left(\frac{|\psi_k|}{\Theta} \right)^2 \cdot dx \tag{A27}
 \end{aligned}$$

which is proportional to $|\psi_k|^2$, thus completing the proof.

Acknowledgments

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