A Study of Bianchi Type IX Spacetime via Time Dependent Quasi-Maxwell Formalism

Morteza Yavari[1]
Department of Physics, Islamic Azad University, Kashan Branch, Kashan, Iran

The exact solutions of the Einstein field equations for the Bianchi type IX metric are obtained via time dependent quasi-Maxwell[2] equations when the matter is a perfect fluid.

1. Introduction

The slicing and threading methods were respectively introduced by Misner, Thorne and Wheeler in 1973 [1] and by Landau and Lifshitz in 1975 [2]. Both methods can be traced back to Landau and Lifshitz [3] in 1941 when they introduced the threading method of splitting the spacetime metric, and in the stationary case the connection, to yield the spatial gravitational force. After them, Lichnerowicz [4] introduced the beginnings of slicing point of view. Also, Møller [5] discussed the spatial gravitational force for a general spacetime. In 1956, Zel'manov [6] discussed the splitting of Einstein field equations in general case. For more details about these formalisms, see reference [7].

It is well known that the Bianchi type cosmological models in presence of perfect fluid play a vital role in general relativity to discuss the early stages of evolution of universe. Also, the Bianchi models can be coupled to any gravitational theory. The Bianchi type IX spacetime is important because FRW with positive curvature, de Sitter and Taub-NUT spacetimes etc. correspond to this spacetime. In this paper, we are going to discuss the TQM equations in threading decomposition formalism for the Bianchi type IX spacetime.

2. TQM Equations

A stationary spacetime[3] $(M, \mathrm{g}_{\mu\nu})$ is a 4-dim Lorentzian manifold with a timelike Killing vector field η^μ. We consider the observers in this spacetime having the velocity components $\lambda^\mu = \frac{\eta^\mu}{\eta}$ in η^μ direction, where $\eta = \sqrt{\mathrm{g}_{\mu\nu}\eta^\mu\eta^\nu}$. In projection formalism [2,8], the metric is decomposed as

$$ds^2 = (\lambda_\mu dx^\mu)^2 + (\mathrm{g}_{\mu\nu} - \lambda_\mu\lambda_\nu)dx^\mu dx^\nu \tag{1}$$

If we choose $\{\eta^\mu\} = (1,0,0,0)$ and $\{\lambda^\mu\} = (\frac{1}{\sqrt{h}}, 0, 0, 0)$, where h is a function of x^μ, then the metric takes the following form [2,8,9]:

$$ds^2 = h(dt - \mathrm{g}_i dx^i)^2 - \gamma_{ij}dx^i dx^j \tag{2}$$

Where, $\gamma_{ij} = -\mathrm{g}_{ij} + h\mathrm{g}_i\mathrm{g}_j$ in which $\mathrm{g}_i = -\dfrac{\mathrm{g}_{0i}}{h}$ and $h = \mathrm{g}_{00}$. It is interesting to rewrite the Einsteins field equations in terms of gravitoelectromagnetism fields[4] in γ-space[5] with time dependent metric γ_{ij}. Therefore, the field equations can be written as the TQM equations[6] [6,12]:

$$^*\nabla\cdot{}^*\mathbf{E} = {}^*\mathrm{E}^2 + \frac{1}{2}{}^*\mathrm{B}^2 - \frac{^*\partial \mathrm{D}}{\partial t} - d - \frac{1}{2}(\zeta + \mathrm{U}) \tag{3}$$

$$^*\nabla\times{}^*\mathbf{B} = 2({}^*\mathbf{S} + {}^*\mathbf{M} - \mathbf{j}_m) \tag{4}$$

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[1] yavarimorteza@yahoo.com
[2] Henceforth abbreviated as TQM.
[3] Greek indices run from 0 to 3 and Latin indices from 1 to 3.

[4] See reference [10] for a discussion of this point.
[5] The quotient space obtained by quotienting spacetime by the action of stationary isometry and it represents the collection of orbits of Killing vectors η^μ, [11].
[6] The symbols () and [] represent the commutation and anticommutation over indices, gravitational units with c=G=1 are used and the 3-dim Levi-Civita tensor ε_{ijk} is antisymmetric under interchange of any pair of indices such that $\varepsilon_{123} = \varepsilon^{123} = 1$, [2]. Also, we note that ${}^*\mathrm{E}_g^2 = \gamma^{ij}\,{}^*\mathrm{E}_{gi}{}^*\mathrm{E}_{gj}$.

$$
\begin{aligned}
{}^*\mathrm{K}_{ij} &= -{}^*\nabla_{(i}{}^*\mathrm{E}_{j)} + {}^*\mathrm{E}_i{}^*\mathrm{E}_j \\
&+\frac{1}{2}({}^*\mathrm{B}_i{}^*\mathrm{B}_j - \gamma_{ij}{}^*\mathrm{B}^2) \\
&+ 2\mathrm{D}_{ik}\mathrm{D}_j^k - \mathrm{D}\mathrm{D}_{ij} + \sqrt{\gamma}\,\varepsilon_{nk(i}\mathrm{D}_{j)}^n{}^*\mathrm{B}^k \\
&- \frac{{}^*\partial \mathrm{D}_{ij}}{\partial t} + \mathrm{U}_{ij} + \frac{1}{2}\gamma_{ij}(\zeta - \mathrm{U}) \qquad (5)
\end{aligned}
$$

Where, $\zeta = \dfrac{\mathrm{T}_{00}}{h}$ is density of the moving matter, $\mathrm{j}^i_m = \dfrac{\mathrm{T}^i_0}{\sqrt{h}}$ is the momentum density, $\mathrm{U}_{ij} = \mathrm{T}_{ij}$ is 3-dim kinematic stress tensors and $\mathrm{U} = \mathrm{U}^i_i$, while $\mathrm{T}_{\mu\nu}$ are energy-momentum tensors. Also, $\dfrac{{}^*\partial}{\partial t} = \dfrac{1}{\sqrt{h}}\dfrac{\partial}{\partial t}$, $\gamma = \det(\gamma_{ij})$ and $d = \mathrm{D}_{ij}\mathrm{D}^{ij}$ such that

$$
\begin{aligned}
\mathrm{D}_{ij} &= \frac{1}{2}\frac{{}^*\partial\gamma_{ij}}{\partial t},\ \mathrm{D}^{ij} = -\frac{1}{2}\frac{{}^*\partial\gamma^{ij}}{\partial t}, \\
&\mathrm{D} = \gamma^{ij}\mathrm{D}_{ij} = \frac{{}^*\partial \ln\sqrt{\gamma}}{\partial t} \qquad (6)
\end{aligned}
$$

and time dependent gravitoelectromagnetism fields are defined in terms of gravoelectric potential $\phi = \ln\sqrt{h}$ and gravomagnetic vector potential $\mathrm{g} = (\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3)$ as follows[7]

$$
{}^*\mathbf{E} = -{}^*\nabla\phi - \frac{\partial \mathrm{g}}{\partial t}\ ;\ {}^*\mathrm{E}_i = -\phi_{*i} - \frac{\partial \mathrm{g}_i}{\partial t} \qquad (7)
$$

$$
\frac{{}^*\mathbf{B}}{\sqrt{h}} = {}^*\nabla \times \mathrm{g}\ ;\ \frac{{}^*\mathrm{B}^i}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\mathrm{g}_{[k*j]} \qquad (8)
$$

In Eqn. (5), ${}^*\mathrm{K}_{ij}$ is 3-dim starry Ricci tensor constructed from 3-dim starry Christoffel symbols as ${}^*\mathrm{K}_{ij} = {}^*\lambda^k_{ij*k} - {}^*\lambda^k_{ik*j} + {}^*\lambda^n_{ij}{}^*\lambda^k_{kn} - {}^*\lambda^n_{ik}{}^*\lambda^k_{nj}$ where ${}^*\lambda^i_{jk} = \frac{1}{2}\gamma^{il}(\gamma_{jl*k} + \gamma_{kl*j} - \gamma_{jk*l})$ and also the starry covariant derivatives of an arbitrary 3-vector and a tensor are given by ${}^*\nabla_j \mathrm{A}_i = \mathrm{A}_{i*j} - {}^*\lambda^k_{ij}\mathrm{A}_k$, ${}^*\nabla_j\mathrm{A}^i = \mathrm{A}^i_{*j} + {}^*\lambda^i_{jk}\mathrm{A}^k$ and ${}^*\nabla_k\mathrm{T}^{ij} = \mathrm{T}^{ij}_{*k} + {}^*\lambda^i_{nk}\mathrm{T}^{jn} + {}^*\lambda^j_{nk}\mathrm{T}^{in}$. Finally, the vectors ${}^*\mathbf{S} = {}^*\mathbf{E} \times {}^*\mathbf{B}$ and $\mathbf{M}$ have components as ${}^*\mathrm{S}^i = \dfrac{\varepsilon^{ijk}}{\sqrt{\gamma}}{}^*\mathrm{E}_j{}^*\mathrm{B}_k$ and ${}^*\mathrm{M}^i = -{}^*\nabla_j\mathrm{D}^{ij} + {}^*\partial^i\mathrm{D}$ in which ${}^*\partial^i = \gamma^{ik}\,{}^*\partial_k$.

[7] Note that the divergence and curl of an arbitrary vector in γ-space are defined by ${}^*\nabla\cdot\mathbf{A} = \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}\,\mathrm{A}^i)_{*i}$ and $({}^*\nabla\times\mathbf{A})^i = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\mathrm{A}_{[k*j]}$ while ${}_{*i} = {}^*\partial_i = \partial_i + \mathrm{g}_i\frac{\partial}{\partial t}$.

2.1. Exact solution of the Bianchi type IX metric via TQM equations

We consider the Bianchi type IX metric [13] as

$$
ds^2 = dt^2 - a^2(dx^2 + dy^2 + dz^2 - 2\cos y\,dxdz) \qquad (9)
$$

Where, a is a function of t. Firstly, it is easy to check that the all components of gravitoelectromagnetism fields are zero and also the nonzero starry Christoffel symbols are

$$
\begin{aligned}
{}^*\lambda^1_{12} &= {}^*\lambda^3_{23} = \frac{1}{2}\cot y, \\
{}^*\lambda^1_{23} &= {}^*\lambda^3_{12} = \frac{1}{2}\csc y, \qquad (10) \\
{}^*\lambda^2_{13} &= -\frac{1}{2}\sin y
\end{aligned}
$$

Now, applying these symbols, we determine the 3-dimensional starry Ricci tensor as

$$
{}^*\mathrm{K}_{ij} = \frac{1}{2}\begin{cases} 1 & i = j = 1, 2, 3, \\ -\cos y & i \neq j = 1, 3, \\ 0 & \text{otherwise} \end{cases} \qquad (11)
$$

In the next step, we assume that the matter content is a perfect fluid, i.e.,

$$
\mathrm{T}_{\mu\nu} = (\rho + p)u_\mu u_\nu - p\mathrm{g}_{\mu\nu} \qquad (12)
$$

Where, ρ, p and u_μ are, respectively, the matter density, pressure and 4-velocity vector of the matter distribution with co-moving coordinates as $u_\alpha = (1, 0, 0, 0)$. Before continuing, we will need to use the following relations[8]

$$
\begin{aligned}
{}^*\mathrm{M} &= 0 \qquad (13) \\
\mathrm{U} &= 3p \qquad (14) \\
\mathrm{U}_{ij}\mathrm{D}^{ij} &= \mathrm{D}p \qquad (15) \\
d = \frac{1}{3}\mathrm{D}^2 &= 3\left(\frac{\dot{a}}{a}\right)^2 \qquad (16)
\end{aligned}
$$

With the help of expression (13), it can be shown that the Eqn. (4) is trivial. Next, with substitution of Eqns. (11), (12), (14) and (16) into Eqns. (3) and (5), respectively,we obtained the following equalities, respectively

$$
\rho + 3p + 6\frac{\ddot{a}}{a} = 0 \qquad (17)
$$

$$
-\rho + p + 2\frac{\ddot{a}}{a} + 4\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = 0 \qquad (18)
$$

[8] Here, dot stand for partial differentiation with respect to t.

Where, physical quantities ρ and p depend on t and y. At this stage, we need the equations of the energy law [6] as follows

$$^*\nabla\cdot\mathbf{j}_m+\frac{^*\partial\zeta}{\partial t}+\mathrm{D}\zeta+\mathrm{U}_{ij}\mathrm{D}^{ij}-2\mathrm{j}_m^k{}^*\mathrm{E}_k=0 \tag{19}$$

$$\frac{^*\partial\,\mathbf{j}_m}{\partial t}+\mathrm{D}\mathbf{j}_m-\zeta^*\mathbf{E}-\mathbf{j}_m\times{}^*\mathbf{B}+\mathbf{\Pi}=0 \tag{20}$$

here $\Pi^i={}^*\nabla_k\mathrm{U}^{ik}-{}^*\mathrm{E}_k\mathrm{U}^{ik}$. In continuation, we can calculate this term as follows[9]

$$\Pi^i=\begin{cases}\dfrac{p'}{a^2} & i=2,\\ 0 & i\neq 2.\end{cases} \tag{21}$$

With the help of this expression, Eqn. (20) changed to

$$p'=0 \tag{22}$$

which is equivalent to $p=p(t)$. Furthermore, with using Eqns. (15) and (16), Eqn. (19) will transform into

$$\dot\rho+(\rho+p)\mathrm{D}=0 \tag{23}$$

To solve the above equation, we assume that ρ to be separable form as follows

$$\rho=f_1(t)+f_2(y) \tag{24}$$

Now, from Eqns. (22)-(24), becomes

$$f_2(y)=0 \tag{25}$$

it is clear that $\rho=\rho(t)$. Finally, with applying Eqns. (16)-(18), we can conclude Eqn. (23) is trivial.

3. Physical Models and Solutions

We now assume that the pressure and matter density of fluid are related through the gamma-law equation of state

$$p=(\gamma-1)\rho \tag{26}$$

Where, γ is the adiabatic parameter and has been taken in the interval $1\le\gamma\le 2$. Below, we will discuss the three physical models corresponding to the $\gamma=1,\frac{4}{3},2$.

<u>Case 1.</u> Dust distribution model (i.e., $\gamma=1$)

In this case, from Eqns. (17) and (18), we find that

$$8a\ddot a+4(\dot a)^2+1=0 \tag{27}$$

The solution of this equation is

$$2\sqrt{\ell a-a^2}-\ell\arctan\left(\frac{2a-\ell}{2\sqrt{\ell a-a^2}}\right)=\pm(t+t_0) \tag{28}$$

Where, ℓ and t_0 are constant. In this case, we can not express term $a(t)$ explicity in terms of t and consequently the physical parameters can not be determined in terms of t. Hence, no physical conclusion can be drawn from this solution.

<u>Case 2.</u> Radiating model (i.e., $\gamma=\frac{4}{3}$)

In this case, from Eqns. (17) and (18), we have

$$4a\ddot a+4(\dot a)^2+1=0 \tag{29}$$

It can be shown that the exact solution of this equation is of the form

$$a=\pm\frac{1}{2}\sqrt{8(c_1-c_2t)-t^2} \tag{30}$$

Where, c_1 and c_2 are constants with conditions

$$c_1,c_2<0\ \ \&\ \ |c_2|>\sqrt{-\frac{c_1}{2}} \tag{31}$$

So, the physical quantities take the following form

$$\rho=3p=\frac{q}{a^4} \tag{32}$$

here $q=3(c_2^2+\frac{c_1}{2})$ is a constant.

<u>Case 3.</u> Zeldovich fluid model (i.e., $\gamma=2$)

In this case, from Eqns. (17) and (18), we have

$$2a\ddot a+4(\dot a)^2+1=0 \tag{33}$$

and it's solution is given by[10]

$$\mathrm{EllipticE}\left(\frac{a}{c},\mathrm{I}\right)-\mathrm{EllipticF}\left(\frac{a}{c},\mathrm{I}\right)=\pm\frac{t+t_0'}{2c} \tag{34}$$

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[9] The over head prime indicate partial differentiation with respect to y.

[10] For more details about Elliptic integrals, see references [14] and [15].

Where, c and t'_0 are constants. Therefore, we cannot obtain term $a(t)$ explicity in terms of t and so the physical parameters cannot be calculated in terms of t.

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