Relativistic and Non-relativistic Solutions of the Inversely Quadratic Yukawa Potential

C. A. Onate*
Theoretical Physics Section, Department of Physics, University of Ilorin, Ilorin, Nigeria

Using the concept of supersymmetric quantum mechanics, we find the relativistic and non-relativistic solutions for the inversely quadratic Yukawa potential. We first find the Hamiltonian of the corresponding Schrödinger equation and then using shape invariance approach we obtain the bound state solutions for both Klein-Gordon and Schrödinger equations and their corresponding eigen-functions. We obtained the solutions of these equations for angular momentum $l \neq 0$ by using Pekeris approximation to overcome the orbital centrifugal barrier.

1. Introduction

Supersymmetry is a mathematical symmetry that relates elementary particles of one spin to other particles which differ by half a unit of spin and are known as super-partners. The concept was first proposed in the context of hadronic physics by Hironari Miyazawa in 1966 whose work was ignored at that time [1-4]. In a theory with unbroken supersymmetry, for every type of boson, there exists a corresponding type of fermion with the same mass and internal quantum numbers and vice versa. However, there is no direct evidence for the existence of supersymmetry [5]. It is motivated by possible solutions to several theoretical problems since the super-partner of the Standard Model particles have not been observed. The supersymmetric partners which appear with masses much greater than 1 TeV are considered the most interesting by particle theorists.

In the application, supersymmetry offers an extension to more familiar symmetries of quantum field theory. The mathematical structure of supersymmetry has subsequently been applied successfully to other areas of physics by Wess, Zumino and Abdus Salam. It remains a vital part of many proposed theories of physics.

There is an increase in the study of both Klein-Gordon and Dirac equations due to the physical importance of their exact solutions, especially of the relativistic equations for the study of systems under certain potentials such as Rosen-Morse, Coulomb, Pöschl-Teller, non-spherical harmonic oscillator, exponential-type potentials [6-13], and others. In these equations, several authors equate the scalar potential to vector potential and obtained the bound state solution for $l \neq 0$ for certain typical potentials [14-16].

In this work, we attempt to study approximate solutions of Schrödinger and Klein-Gordon equations with an inversely quadratic Yukawa potential given by [17]

$$V(r) = -\frac{V_0}{r^2} e^{-2\alpha r} \tag{1}$$

Where, α is the screening parameter and V_0 is the depth of the potential. To obtain the solution of above mentioned two equations for angular momentum $l \neq 0$, we apply a suitable approximation type given by [18]

$$\frac{1}{r^2} = \frac{4\alpha^2}{(1-e^{-2\alpha r})^2} e^{-2\alpha r} \tag{2}$$

2. The Klein-Gordon Equation with Scalar and Vector Potentials

The Klein-Gordon equation is written as

$$\left[\frac{d^2}{dr^2} + \left(M + S(r)\right)^2 - \left(E - V(r)\right)^2 - \frac{l(l+1)}{r^2}\right] R(r) = 0 \tag{3}$$

A critical investigation by Alhaidari et al. [16], shows that $S = \pm V$.

Since $S = \pm V$, the equation describes a scalar particle (spin-0 particle). This is the Schrödinger equation for the potential 2V in the non-relativistic limit. A conclusion was drawn by Alhaidari et al. that the only choice is $S = +V$, which results into a

*onateca12@gmx.us

nontrivial non-relativistic limit with potential function 2V and not V [19]. Thus in the relativistic limit, the interaction potential becomes V instead of 2V and then we write our Klein-Gordon equation as

$$\left[\frac{d^2}{dr^2}-\left(E_{n,l}-\frac{1}{2}V(r)\right)^2\right.$$

$$\left.+\left(M+\frac{1}{2}S(r)\right)^2-\frac{l(l+1)}{r^2}\right]R_{n,l}(r)=0 \qquad (4)$$

3. Bound State Solution of the Klein-Gordon Equation

The Klein-Gordon equation with scalar potential $S(r)$ and vector potential $V(r)$ [20,21] is given by

$$\frac{d^2U_{nl}(r)}{dr^2}+(E^2-M^2-V(r)(E_{n,l}+M)$$

$$-\frac{l(l+1)}{r^2})U_{nl}(r)=0 \qquad (5)$$

With Eqns. (1) and (2), Eqn. (5) becomes

$$\frac{d^2U_{nl}(r)}{dr^2}=\left[V_{eff}-\tilde{E}_{nl}\right]U_{nl}(r) \qquad (6)$$

Where we have made the following substitutions

$$V_{eff}=\frac{B_1e^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{B_1e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} \qquad (7a)$$

$$B_1=-4V_0\alpha^2(E_{nl}+M) \qquad (7b)$$

$$B_2=4\alpha^2(l(l+1)-2V_0) \qquad (7c)$$

$$\tilde{E}_{nl}=E_{nl}^2-M^2 \qquad (7d)$$

Writing the ground-state wave function $U_{0,l}(r)$ [22,23] as

$$U_{0,l}(r)=\exp\left(-\int\psi(r)dr\right) \qquad (8)$$

Where, $\psi(r)$ is the supersymmetric super-potential. Substituting Eqn. (8) into Eqn. (6), we have a second order differential equation in the form of Riccati equation, which is satisfied by the super-potential $\psi(r)$,

$$\psi^2(r)-\frac{d\psi(r)}{dr}=V_{eff}-\overline{E}_{nl} \qquad (9)$$

Where

$$\psi(r)=\rho+\frac{ðe^{-2\alpha r}}{1-e^{-2\alpha r}}\ . \qquad (10)$$

Here, $E_{0,l}$ represents the ground-state energy, ρ and C are two constants.

The wave function related to the ground state is now obtained from the solution of the Riccati equation (Eqn. (9)) as

$$\rho=-(\frac{ð^2+B_1}{2ð})\ , \qquad (11)$$

And

$$ð=-\alpha\pm\sqrt{\alpha^2+B_2} \qquad (12)$$

By using the super-potential function of Eqn. (16), the partner potentials of the inversely quadratic Yukawa potential for supersymmetry quantum mechanics are then given as

$$U_+(r)=\psi^2+\psi'=\rho^2+\frac{2\rho ðe^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{ðe^{-2\alpha r}(ðe^{-2\alpha r}+2\alpha)}{\left(1-e^{-2\alpha r}\right)^2}$$

$$=\frac{ðe^{-2\alpha r}\left(ð+2\alpha\right)+ð^2e^{-2\alpha r}(e^{-2\alpha r}-1)}{\left(1-e^{-2\alpha r}\right)^2}+\frac{2\rho ðe^{-2\alpha r}}{1-e^{-2\alpha r}}-\left(\frac{ð^2+B_2}{2ð}\right)^2 \qquad (13)$$

$$U_-(r)=\psi^2-\psi'=\rho^2+\frac{2\rho ðe^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{ðe^{-2\alpha r}(ðe^{-2\alpha r}-2\alpha)}{\left(1-e^{-2\alpha r}\right)^2}$$

$$=\frac{ðe^{-2\alpha r}\left(ð-2\alpha\right)+ð^2e^{-2\alpha r}(e^{-2\alpha r}-1)}{\left(1-e^{-2\alpha r}\right)^2}+\frac{2\rho ðe^{-2\alpha r}}{1-e^{-2\alpha r}}-\left(\frac{ð^2+B_2}{2ð}\right)^2\ . \qquad (14)$$

From Eqns. (13) and (14), we write

$$R(a_1)=U_+(r,a_0)-U_-(r,a_1) \qquad (15)$$

From Eqn. (15), the shape invariance condition is satisfied and the shape invariance holds via mapping of the form [24-29] $C \to C - 2\alpha$. Now, consider our Eqn. (11), it is easy to write

$$E_{n,l} = -\left(\frac{ð^2 + B_1}{2ð}\right)^2, \tag{16}$$

Where, $a_n = ð - \alpha n$, which on substitution gives the energy equation as

$$E_{nl}^2 - M^2 + \alpha^2 \left[\frac{\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{(1+2l)^2 - 4V_0(E_{nl} + M)}\right)^2 + V_0(E_{nl} + M)}{n + \frac{1}{2} + \frac{1}{2}\sqrt{(1+2l)^2 - 4V_0(E_{nl} + M)}}\right]^2 = 0 \tag{17}$$

Now, let us obtain the non-relativistic limit by making a transformation as follows: $E_{n,l} + M = \frac{2\mu}{\hbar^2}$ and $E_{n,l} - M = E_{n,l}$. With these transformations, the non-relativistic limit on Eqn. (17) is obtain as

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{(1+2l)^2 - \frac{8\mu V_0}{\hbar^2}}\right)^2 + \frac{2\mu V_0}{\hbar^2}}{n + \frac{1}{2} + \frac{1}{2}\sqrt{(1+2l)^2 - \frac{8\mu V_0}{\hbar^2}}}\right]^2 \tag{18}$$

Now, we can obtain the corresponding un-normalized wave functions via the standard function analysis method. Let us now define a new variable of the form $y = e^{-\alpha r}$ and substituting it into Eqn. (6) results to the following second order differential equation

$$\frac{d^2R}{dy^2} + \frac{1}{y}\frac{dR}{dy} + \left[\frac{A + By + Cy^2}{y^2(1-y)^2}\right]R = 0 \tag{19}$$

Where

$$A = \frac{\mu E_{nl}}{2\alpha^2\hbar^2}, \qquad B = -\frac{\mu E_{nl}}{\hbar^2\alpha^2} - \ell(\ell+1),$$

$$C = \frac{2\mu V_0}{\hbar^2} + \frac{\mu E_{nl}}{\hbar^2\alpha^2} \tag{20}$$

Now, from our transformation, we can write

$$R_{nl}(y) = y^{\varepsilon}(1-y)^{\eta}F_{nl}(y) \tag{21}$$

Where, $\varepsilon = \left(\frac{\mu E_{nl}}{2\alpha^2\hbar^2}\right)^{\frac{1}{2}}$ and $\eta = \frac{1 + \left((1+2l)^2 - \frac{8\mu V_0}{\hbar^2}\right)^{\frac{1}{2}}}{2}$. Now, let $\xi = \left(\frac{-\mu E_{nl}}{2\alpha^2\hbar^2}\right)^{\frac{1}{2}}$ Eqn. (19) then reduces to a second-order homogeneous linear differential equation of the form

$$F''(y) + F'(y)\left[\frac{(2\xi+1) - y(2\xi+\eta+1)}{y(1-y)}\right] - F(y)\left[\frac{(2\xi+\eta)^2 + \left(\frac{\mu E_{nl}}{\alpha^2\hbar^2}l(l+1)\right)}{y(1-y)}\right] = 0 \tag{22}$$

And consequently, the total radial wave function is obtained as

$$R_{nl}(y) = N_{nl} e^{-\alpha} \xi (1-y)^{\eta} \, {}_2F_1(-n, n+2(\xi+\eta)+2\xi+1, y) \tag{23}$$

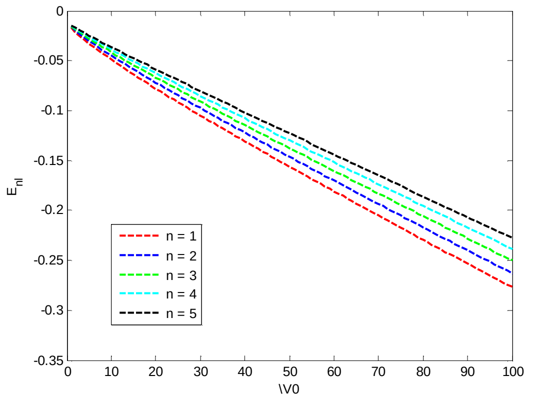

Fig.1: $E_{n,l}$ vs V_0 for $\mu = \hbar = l = 1, \alpha = 0.15$.

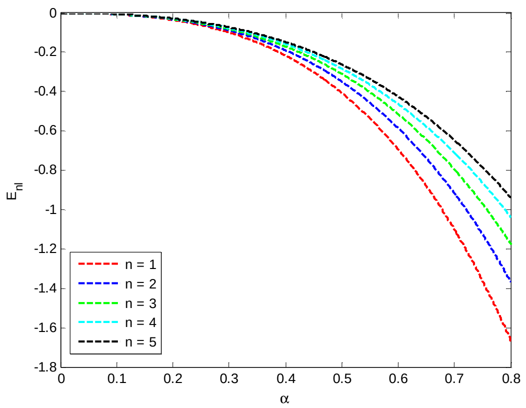

Fig.2: $E_{n,l}$ vs α for $V_0 = 1$, $\mu = \hbar = 1$ and $l = 0$.

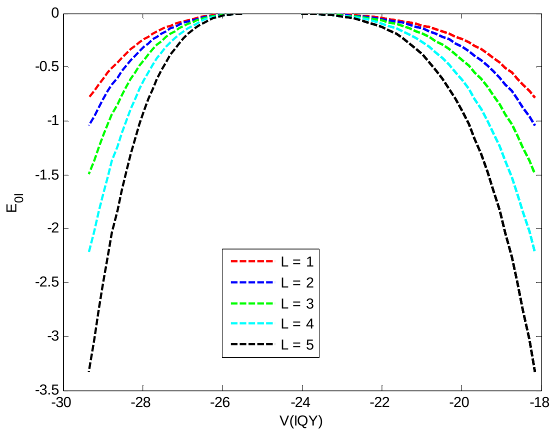

Fig.3: $E_{n,0}$ vs $V_{IQY}(r)$ for $n = V_0 = \hbar = \mu = 1$.

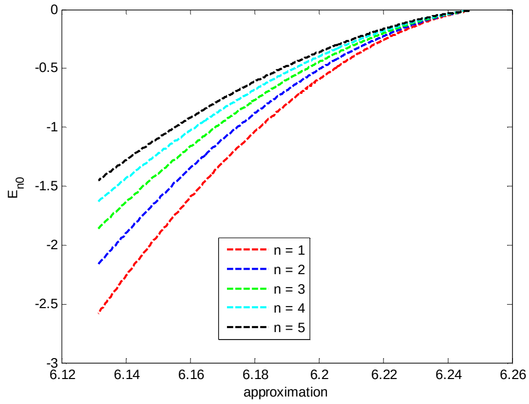

Fig.4: E_{n0} vs approximation for $\mu = \hbar = V_0 = 1$ and $l = 0$.

4. Some Expectation Values for Inversely Quadratic Yukawa Potential

We calculate some expectation values of this potential using Hellmann-Feynman theorem [30-33]. If the Hamiltonian H for a particular quantum system is a function of the parameter V_0, then taking the energy eigen-values as $E_{n,l}(V_0)$ and the eigen-functions as $U_{n,l}(V_0)$ of the Hamiltonian, we can easily write from the HFT

$$\frac{\partial E_{n,l}(V_0)}{\partial V_0} = \langle U_{n,l}(V_0) \left| \frac{\partial H(V_0)}{\partial V_0} \right| U_{n,l}(V_0) \rangle \tag{24}$$

Where the effective Hamiltonian is given by

$$H = \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} - \frac{V_0 e^{-2\alpha r}}{r^2} \tag{25}$$

Then, we obtain

$$\frac{\partial E_{n,l}}{\partial V_0} = \langle -\frac{e^{-2\alpha r}}{r^2} \rangle \tag{26}$$

5. Conclusion

We have obtained the energy eigenvalue equation for Klein-Gordon equation and its non-relativistic limit (Schrödinger equation) using SUSY QM formalism and methodology for angular momentum $l \neq 0$ using a suitable approximation scheme. We have also obtained the corresponding eigen-function as well as the expectation value.

References

[1] H. Miyazawa, Prog. Theor. Phys. **36**, 1266 (1966).
[2] H. Miyazawa, Phys. Rev. **170**, 1586 (1968).
[3] Michio Kaku, Quantum Field Theory, ISBN 0-19-509158-2.
[4] Peter Freund, Introduction to Supersymmetry, ISBN 0-521-35675-x.
[5] Gordon L. Kane, The Dawn of Physics Beyond the Standard Model, Scientific American, June (2003).
[6] A. deSouzaDutra and G. Chen, Phys. Lett. A **349**, 297 (2006).
[7] C. Rojas and V. M. Villalba, Phys. Rev. A. **71**, 052101 (2005).
[8] A. Dominguez, Phys. Lett. A **136**, 175 (1989).
[9] A. Z. Zhang, K. Chen and S. D. Sun, Chin. Phys. **14**, 42 (2005).
[10] H. S. Dong, G. H. Sun and M. Lozada-Cassou, Phys. Lett. A **328**, 299 (2004).
[11] M. V. Villalba and C. Rojas, Int. Mod. Phys. A **21**, 313 (2006).
[12] F. Yasuk, C. Berkdemir and A. J. Berkdemir, Phys. A **38**, 6579 (2005).
[13] F. L. Lu, C. Y. Chen and D. S. Sun, Chin. Phys. **14**, 463 (2005).
[14] F. Yasuk, A. Durmus and I. Boztosun, J. Math. Phys. **47**, 082302 (2006).
[15] L. Z. Yi, Y. F. Diao, J. Y. Liu and C. S. Jia, Phys. Lett. A **333**, 212 (2004).
[16] A. Alhaidari, H. Bahlouli and A. Al-Hassan, Phys. Lett. A **349**, 87 (2006).
[17] M. Hamzavi, S. M. Ikhdair and B. I. Ita, Phys. Scr. **85**, 045009 (2012).
[18] E. Maghsoodi, H. Hassanabadi and O. Aydoğdu, Phys. Scr. **86**, 015005 (2012).
[19] D. Aysen and Y. Fevziye, J. Chem. Phys. **120**, 074108 (2007).
[20] B. J. Falaye, J. Math. Phys. **53**, 082107 (2012).
[21] Q. Wen-Chao and D. Shi-Hai, Phys. Lett. A **372**, 4789 (2008).
[22] Z. Lie-Hui, L. Xiao-Ping and J. Chun-Sheng, Int. J. Quant. Chem. **111**, 1870 (2011).
[23] K. J. Oyewumi and C. O. Akoshile, Eur. Phys. J. A **45**, 311 (2010).
[24] H. Hassanabadi, E. Maghsoodi and S. Zarrinkamar, Eur. Phys. J. Plus **127**, 31 (2012).
[25] S. Jia Chun, Y. Zhang, L. Zeng Xiang and Sun T. Liang, Commun. Theor. Phys. **36**, 611 (2001).
[26] S. Zarrinkamar, A. A. Rajabi and H. Hassanabadi, Phys. Scr. **83**, 015009 (2011).
[27] S. Zarrinkamar, A. A. Rajabi and H. Hassanabadi, Ann. Phys. **325**, 2522 (2010).
[28] S. Zarrinkamar, H. Hassanabadi and A. A. Rajabi, Int. J. Mod. Phys. A **26**, 1011 (2011).
[29] C. A. Onate, K. J. Oyewumi and B. J. Falaye, The African Rev. Phys. **8**, 129 (2013).
[30] R. P. Feynman, Phys. Rev. **56**, 340 (1939).
[31] G. Hellmann, Introduction to Quantum Chemistry (1937) (in German).
[32] D. Agboola, Phys. Scr. **80**, 065304 (2009).
[33] H. Hassanabadi, L. L. Lu, S. Zarrinkamar, G. Liu and H. Rahimov, Acta Physica Polonica A **122**, 1111 (2012).

Received: 30 August, 2013
Accepted: 25 October, 2013