Regular and Semi-Regular 4D-Polytopes of the Coxeter-Weyl Group $W(SO(9))$ and Quaternions

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The Coxeter-Weyl group $W(SO(9))$ is constructed in terms of quaternions and its orbits representing the vertices of the 4D-polytopes corresponding to the Platonic and Archimedean versions of polyhedra in three dimensions are determined. The vertices of the polytopes are displayed in terms of discrete quaternions projected to three dimensions using its Coxeter-Weyl subgroups $W(SO(7))$ and $W(SU(4))$ which are the respective symmetries of the solids possessing octahedral and tetrahedral symmetries. We also explain the cell structures of the 15 polytopes of interest and construct the vertices with the use of all subgroups $W(SO(7))$, $W(SU(4))$, $D_4 \times Z_2$, $D_3 \times Z_2$ of $W(SO(9))$ acting in three dimensions.

1. Introduction

The Platonic solids, tetrahedron, cube, octahedron, icosahedron and dodecahedron have been known since the neolithic time [1]. The Archimedean polyhedra are solids which are obtained from the Platonic solids through rectifications and truncations. In a previous paper we have constructed the vertices of the Platonic and Archimedean solids in terms of quaternions using the quaternionic representations of the Coxeter-Weyl groups $W(SU(4))$, $W(SO(7))$ and $W(H_3)$ [2]. Actually, orbits of the highest weights of the Coxeter-Weyl group representing the irreducible representations of the Lie algebras correspond to the vertices of the polytopes in higher dimensions, provided the short roots are converted to the long roots [3]. The Platonic and Archimedean solids have been successfully applied to describe the crystallography in physics, molecular symmetries in chemistry [4] and some virus structures in biology [5]. The Coxeter groups H_2, H_3 and H_4 and their affine extensions may find direct applications in quasi-crystallography [6].

The rank-4 Lie groups $SO(8)$ and $SO(9)$, subgroups of the exceptional Lie group F_4, are the little groups of the supersymmetric string theories in 10-dimensional space-time and M- theory in 11-dimensional space-time [7] respectively. Certain irreducible representations of $SO(9)$ are used for the classifications of the massless supergravity multiplets. The compactification of M-theory may require the Lie groups $SO(7)$ and G_2 as holonomy groups [8]. The Coxeter-Weyl groups of rank-4 Lie algebras are important for the determination of the irreducible representations of the concerned Lie groups. It was noted before that the Coxeter-Weyl groups $W(SO(8))$, $W(SO(9))$ and $W(F_4)$ can be constructed as finite subgroups of $O(4)$ that can be classified as direct products of finite subgroups of quaternions [9], isomorphic to the finite subgroups of $SU(2)$.There exists a natural correspondence between the finite subgroups of $O(4)$ and the Coxeter-Weyl groups of the rank-4 Lie algebras. For example, the automorphism group $Aut(F_4)$ can be represented as $Aut(F_4) \approx \{[O,O] \oplus [O,O]^*\}$ which represents the direct product of two binary octahedral groups acting on a quaternion and its conjugate. In this paper, we first discuss the construction of the group $W(SO(9))$ and study its orbits in terms of quaternions. We show that orbits of highest weights of certain irreducible representations correspond to regular and semi-regular polytopes in four dimensions. One can find discussions on regular polytopes of $W(SO(9))$ in the references [10, 11]. The classification of regular and semi-regular polytopes of $W(SO(9))$ with the cell structures can be found in references [12]. Our construction of polytopes is based on Lie algebraic technique with the quaternionic representation of the Coxeter-Weyl group $W(SO(9))$. The paper is organized as follows: In Section 2, we introduce quaternions and its finite groups relevant for the construction of $W(SO(9))$. We study the Coxeter-

Weyl orbit of $W(SO(9))$ for a general weight vector $\Lambda = (a_1 a_2 a_3 a_4)$, $(a_i \geq 0; i = 1,2,3,4)$ in Section 3 and give decompositions of the orbit under the subgroups $W(SO(7))$, $W(SU(4))$, $D_4 \times Z_2$, $D_3 \times Z_2$ which are relevant for the cell structures of the $W(SO(9))$ polytopes. Section 4 is devoted to the study of regular polytopes. In Section 5, we prove the existence of prismatic polyhedra in certain semi-regular polytopes. Section 6 discusses the individual properties of the semi-regular polytopes of $W(SO(9))$. Section 7 involves the concluding remarks on the use of quaternionic construction of the Coxeter-Weyl group $W(SO(9))$.

2. Quaternionic representation of the Coxeter-Weyl group $W(SO(9))$

Let $q = q_0 + q_i e_j$, $(i = 1,2,3)$ be a real unit quaternion with its conjugate defined by $\bar{q} = q_0 - q_i e_i$ and the norm $q\bar{q} = \bar{q}q = 1$. Here the quaternionic imaginary units satisfy the relations

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k, \ (i,j,k = 1,2,3) \qquad (1)$$

where δ_{ij} and ϵ_{ijk} are the Kronecker and Levi-Civita symbols respectively and summation over the repeated indices is implicit. With the definition of the scalar product

$$(p,q) = \frac{1}{2}(\bar{p}q + \bar{q}p), \qquad (2)$$

quaternions generate the four-dimensional Euclidean space. The group of quaternions is isomorphic to $SU(2)$ which is the double cover of the proper rotation group $SO(3)$. Its finite subgroups are all well known [10,11]. It has an infinite number of cyclic and dicyclic groups in addition to the binary tetrahedral group T, binary octahedral group O and the binary icosahedral group I which are related to the ADE classification of the Lie algebras [13]. An orthogonal rotation in 4D Euclidean space can be represented by the group elements of $O(4)$ [14] as

$$[a,b] : q \to q' = aqb, [c,d]^* : q \to q'' = c\bar{q}d. \qquad (3)$$

where a, b, c, d are unit quaternions and q can be a quaternion with arbitrary norm. The finite subgroups of $O(4)$ follows the finite subgroups of $SU(2)$. The relevant finite subgroup of $SU(2)$ here is the binary octahedral group O which can be decomposed as follows:

$$O = T \oplus T'. \qquad (4)$$

Here T represents the binary tetrahedral group given by

$$T = \{\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\}, \ (5)$$

also representing the quaternionic vertices of the polytope {3,4,3}. The dual polytope is represented by the quaternions

$$T' = \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3), \frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1), \frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\}. \qquad (6)$$

Let $p, q \in O$ be arbitrary elements of the binary octahedral group then the set of elements

$$Aut(F_4) \approx W(F_4) : Z_2 = \{[p,q] \oplus [p,q]^*\} \qquad (7)$$

is the extension of the Coxeter-Weyl group $W(F_4)$ by the diagram symmetry [9]. The Coxeter-Weyl group of order 384 is a maximal subgroup of $W(F_4)$ with index 3. The quaterionic simple roots where the short root is converted to a long root are shown in the Coxeter-Dynkin diagram of $SO(9)$ in Fig. 1.

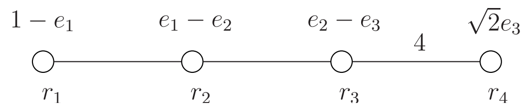


FIG. 1: The Coxeter-Dynkin diagram of $W(SO(9))$.

The reflection generator r of an arbitrary Coxeter group with respect to a hyperplane represented by the vector α is given by the action

$$r : \Lambda \to \Lambda - \frac{2(\Lambda,\alpha)}{\alpha,\alpha}\alpha. \qquad (8)$$

When Λ and α are represented by quaternions the equation (8) reads

$$r : \Lambda \to -\frac{\alpha\bar{\Lambda}\alpha}{(\alpha,\alpha)},$$

and the generators of $W(SO(9))$ can be written in terms of the notation of (3) as

$$\begin{aligned} r_1 &= \frac{1}{2}[(1-e_1),(-1+e_1)]^*, \\ r_2 &= \frac{1}{2}[(e_1-e_2),(-e_1+e_2)]^*, \\ r_3 &= \frac{1}{2}[(e_2-e_3),(-e_2+e_3)]^*, \\ r_4 &= [e_3,-e_3]^*. \end{aligned} \qquad (9)$$

Now we want to give the group elements of $W(SO(9))$ in a compact form. To do this we decompose the elements of the binary octahedral group O into subsets defined by [9]

$$O = T \oplus T' = \{V_0 \oplus V_+ \oplus V_-\} \oplus \{V_1 \oplus V_2 \oplus V_3\} \quad (10)$$

with

$$\begin{aligned} V_0 &= \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}, \\ V_+ &= \frac{1}{2}\{\pm 1 \pm e_1 \pm e_2 \pm e_3\} \\ &\quad \text{(even number of + signs)}, \ V_- = \bar{V}_+ \\ V_1 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1)\}, \ \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)\}, \\ V_2 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_2)\}, \ \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1)\}, \\ V_3 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_3)\}, \ \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\}. \end{aligned} \quad (11)$$

They satisfy the multiplication table given in Table I.

TABLE I: Multiplication table of the binary octahedral group.

	V_0	V_+	V_-	V_1	V_2	V_3
V_0	V_0	V_+	V_-	V_1	V_2	V_3
V_+	V_+	V_-	V_0	V_3	V_1	V_2
V_-	V_-	V_0	V_+	V_2	V_3	V_1
V_1	V_1	V_2	V_3	V_0	V_+	V_-
V_2	V_2	V_3	V_1	V_-	V_0	V_+
V_3	V_3	V_1	V_2	V_+	V_-	V_0

One can show that the elements of $W(SO(9))$ can be written as follows

$$W(SO(9)) \approx A \oplus A^* \oplus B \oplus B^* \quad (12)$$

where

$$\begin{aligned} A &= \{[V_0, V_0] \oplus [V_+, V_-] \oplus [V_-, V_+]\}, \\ B &= \{[V_1, V_1] \oplus [V_2, V_2] \oplus [V_3, V_3]\}. \end{aligned} \quad (13)$$

Elements A^* and B^* are defined in (3). One can use Table I to show that the set of quaternions V_0 and $V_+ \oplus V_-$ are separately left invariant under $W(SO(9))$. In other words, they are the $W(SO(9))$ orbits of sizes 8 and 16, respectively. The orbits V_0 and $V_+ \oplus V_-$ are called the 16-cell and 8-cell respectively, which will be discussed in more detail in Section 4. Here we note one aspect of $W(SO(9))$ that it can be written as the semi-direct product (:) of the elementary abelian group $Z_2^4 = Z_2 \times Z_2 \times Z_2 \times Z_2$ and the symmetric group S_4, that is, $W(SO(9)) \approx Z_2^4 : S_4$. The generators of Z_2^4 and S_4 can be taken respectively as

$$[1, -1]^*, \ [e_1, -e_1]^*, \ [e_2, e_2]^*, [e_3, e_3]^* \quad (14)$$

and

$$\begin{aligned} f &= \frac{1}{2}[(1+e_2), \ \omega_0(1-e_2)\omega_0]^*, \ f^4 = [1,1], \\ g &= [\omega_0, \bar{\omega}_0], \ g^3 = [1,1], \ \omega_0 = \frac{1}{2}(1+e_1+e_2+e_3). \end{aligned} \quad (15)$$

Note that the generators a, b of S_4 permute the generators of the elementary abelian group Z_2^4 by conjugation, therefore, Z_2^4 is an invariant subgroup of the group $W(SO(9))$. The elements of S_4 can be compactly written as [2, 9]

$$S_4 = \{[p, \bar{\omega}_0 \bar{p} \omega_0] \oplus [t, \omega_0 \bar{t} \omega_0]^*\}, \ p \in T, \ t \in T'. \quad (16)$$

It is the Weyl subgroup $W(SU(4)) \approx S_4$ of $W(SO(9))$ generated by the first three generators $r_1, \ r_2, \ r_3$ which leave the quaternion ω_0 invariant. The generators $r_2, \ r_3, \ r_4$ generate the subgroup $W(SO(7)) \approx S_4 \times Z_2$ given by [2,9]

$$W(SO(7)) \approx S_4 \times Z_2 = \{[p, \bar{p}] \oplus [t, \bar{t}] \oplus [p, \bar{p}]^* \oplus [t, \bar{t}]^*\}$$

which leaves the scalar part of the quaternion invariant. There are two more subgroups relevant to the cell structures of the orbits $W(SO(9))$. They are the prismatic groups $D_3 \times Z_2$ and $D_4 \times Z_2$ generated by $(r_1, \ r_2, \ r_4)$ and $(r_1, \ r_3, \ r_4)$ respectively. Here (r_1, r_2) generate the dihedral group $D_3 \approx W(SU(3))$ and Z_2 is generated by r_4. Similarly (r_3, r_4) and r_1 generate the dihedral group D_4 and Z_2, respectively.

3. Orbit of the group $W(SO(9))$ in terms of quaternions

Let $\Lambda = (a_1 \ a_2 \ a_3 \ a_4)$, $(a_i \geq 0, \ i = 1,2,3,4)$ represents a vector with non-negative integer Dynkin labels which characterizes the irreducible representations of the Lie group $SO(9)$ [15] provided the fourth root in Fig. 1 is a short root. In what follows we show that the Coxeter-Weyl orbits of $W(SO(9))$ represent the regular and semi-regular 4D polytopes when the fourth root is taken to be a long root. The highest weight vector can be written as a linear combination of the simple roots of

Fig. 1,

$$\Lambda = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 = (x_1, x_2, x_3, x_4)C \quad (17)$$

where C is the Cartan matrix given by

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} & 2 \end{pmatrix}. \quad (18)$$

The Cartan matrix indicates that the short root is converted to a long root. Equation (17) will determine the vector Λ in terms of quaternionic units in the form $\Lambda = a + be_1 + ce_2 + de_3$. Action of the elements of the elementary abelian group Z_2^4 will lead to the set of 16 quaternions

$$\pm a \pm be_1 \pm ce_2 \pm de_3 \quad (19)$$

with

$$\begin{aligned} a &= a_1 + a_2 + a_3 + \frac{a_4}{\sqrt{2}}, \\ b &= a_2 + a_3 + \frac{a_4}{\sqrt{2}}, \\ c &= a_3 + \frac{a_4}{\sqrt{2}},\ d = \frac{a_4}{\sqrt{2}}. \end{aligned} \quad (20)$$

Since the elements of the symmetric group S_4 permute the quaternionic units $1, e_1, e_2, e_3$ [2] we obtain 24 sets of elements of (19) in the following form

$$\begin{aligned} &\pm a \pm be_1 \pm ce_2 \pm de_3,\ \pm a \pm ce_1 \pm de_2 \pm be_3, \\ &\pm a \pm de_1 \pm be_2 \pm ce_3,\ \pm a \pm be_1 \pm de_2 \pm ce_3, \\ &\pm a \pm de_1 \pm ce_2 \pm be_3,\ \pm a \pm ce_1 \pm be_2 \pm de_3, \\ &\pm b \pm ce_1 \pm de_2 \pm ae_3,\ \pm b \pm de_1 \pm ae_2 \pm ce_3, \\ &\pm b \pm ae_1 \pm ce_2 \pm de_3,\ \pm b \pm ce_1 \pm ae_2 \pm de_3, \\ &\pm b \pm ae_1 \pm de_2 \pm ce_3,\ \pm b \pm de_1 \pm ce_2 \pm ae_3, \\ &\pm c \pm de_1 \pm ae_2 \pm be_3,\ \pm c \pm ae_1 \pm be_2 \pm de_3, \\ &\pm c \pm be_1 \pm de_2 \pm ae_3,\ \pm c \pm de_1 \pm be_2 \pm ae_3, \\ &\pm c \pm be_1 \pm ae_2 \pm de_3,\ \pm c \pm ae_1 \pm de_2 \pm be_3, \\ &\pm d \pm ae_1 \pm be_2 \pm ce_3,\ \pm d \pm be_1 \pm ce_2 \pm ae_3, \\ &\pm d \pm ce_1 \pm ae_2 \pm be_3,\ \pm d \pm ae_1 \pm ce_2 \pm be_3, \\ &\pm d \pm be_1 \pm ce_2 \pm ae_3,\ \pm d \pm be_1 \pm ae_2 \pm ce_3. \end{aligned} \quad (21)$$

This is the orbit of $W(SO(9))$ of size 384 for arbitrary non-negative integers $a_i,\ (i = 1, 2, 3, 4)$. We note that each 16-set of elements is invariant under the group Z_2^4 and the elements of S_4 transform 24 of 16-sets to each other. Now we can discuss the decomposition of the orbit in (21) under the subgroups of $W(SO(7))$ and $W(SU(4))$. The decomposition of $W(SO(9))$ orbit under $W(SO(7))$ is almost straightforward since $W(SO(7))$ leaves the real part of the quaternion invariant. That means we have to classify the quaternions in (21) having the same real component as the orbits of $W(SO(7))$. The coefficients $\pm a, \pm b, \pm c, \pm d$, appear as real components in certain sets of quaternions containing 48 quaternions each. Therefore the $W(SO(9))$ orbit of size 384 decomposes as 8 sets of orbits of size 48 each, $384 = 8 \times 48$. Of course, this is the most general case where all Dynkin labels $a_i,\ (i = 1, 2, 3, 4)$ are taken to be different from zero. For special values of a_i $(i = 1, 2, 3, 4)$, the size of the orbit will be smaller than 48 which will be discussed in the next section. To obtain the decomposition of (21) under $W(SU(4))$ it is better to choose a more suitable base of quaternions. Since $W(SU(4))$ leaves ω_0 invariant, one can define the new orthogonal basis as

$$\begin{aligned} \omega_0 &= \frac{1}{2}(1 + e_1 + e_2 + e_3), \\ \omega_1 &= \frac{1}{2}(1 + e_1 - e_2 - e_3), \\ \omega_2 &= \frac{1}{2}(1 - e_1 + e_2 - e_3), \\ \omega_3 &= \frac{1}{2}(1 - e_1 - e_2 + e_3). \end{aligned} \quad (22)$$

Replacing the unit quaternions $1, e_1, e_2, e_3$ in terms of $\omega_i,\ (i = 0, 1, 2, 3)$ and substituting in (21) one obtains the $W(SO(9))$ orbit expressed in terms of new bases of quaternions $\omega_a,\ (a = 0, 1, 2, 3)$. Then the $W(SO(9))$ orbit will be classified as 24-sets of quaternions with the coefficients of ω_0 taking 16 different values. When all Dynkin labels take values $a_i = 1,\ (i = 1, 2, 3, 4)$ then these 16 coefficients are given by

$$\begin{gathered} \pm(3 + \sqrt{2})\omega_0, \pm(3 + \frac{1}{\sqrt{2}})\omega_0,\ \pm (2 + \frac{1}{\sqrt{2}})\omega_0, \\ \pm 2\omega_0, \pm(1 + \frac{1}{\sqrt{2}})\omega_0,\ \pm \omega_0,\ \pm \frac{1}{\sqrt{2}}\omega_0 \end{gathered} \quad (23)$$

with two more 24-sets of quaternions having zero coefficients of ω_0. This shows that 384 elements of quaternions decompose as $384 = 16 \times 24$ where the largest size of orbit of $W(SU(4))$ is 24.

4. The $W(SO(9))$ orbits as regular polytopes

There are only two regular polytopes of $W(SO(9))$ with the orbit sizes 8 and 16 as we have pointed out in Section 2. They are the sets of quaternions $V_0 = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$ and $V_+ \oplus V_- = \frac{1}{2}\{\pm 1 \pm e_1 \pm e_2 \pm e_3\}$. In the context of Dynkin labels, the set V_0 correspond to the *orbit*-(1000) with 8 quaternions representing a hyperoctahedron. Its dual $V_+ \oplus V_-$ is represented by the *orbit*-(0001) which is a hypercube in 4-dimensional Euclidean space. Before we explain their cell structures, we first note that V_0 decomposes as $V_0 = \{1 \oplus (-1) \oplus (\pm e_1, \pm e_2, \pm e_3)\}$ under $W(SO(7))$ which consists of two single points and one octahedron in three dimensional Euclidean space. Similarly $V_+ \oplus V_-$, decomposes as the sum of two cubes

$$V_+ \oplus V_- = \frac{1}{2}\{1 \pm e_1 \pm e_2 \pm e_3\},\ \frac{1}{2}\{-1 \pm e_1 \pm e_2 \pm e_3\} \tag{24}$$

The two orbits of $W(SO(9))$ decompose under $W(SU(4))$ as follows:

$$\begin{aligned}(orbit\ -\ (1000)) &= V_0 \\ &= \{\frac{1}{2}(\omega_0 \pm \omega_1 \pm \omega_2 \pm \omega_3)(\text{even } (+) \text{ sign}), \\ &\quad \frac{1}{2}(-\omega_0 \pm \omega_1 \pm \omega_2 \pm \omega_3)(\text{odd } (+) \text{ sign})\}.\end{aligned} \tag{25}$$

which correspond to two tetrahedra and the orbit-(0001) consists of two single points, two tetrahedra and one octahedron represented by respective quaternions,

$$\begin{aligned}&\pm\sqrt{2}\omega_0, \frac{1}{\sqrt{2}}(\omega_0 \pm \omega_1 \pm \omega_2 \pm \omega_3)(\text{odd (-) sign})\}, \\ &\quad \frac{1}{\sqrt{2}}(-\omega_0 \pm \omega_1 \pm \omega_2 \pm \omega_3)(\text{odd } (+) \text{ sign})\}, \\ &\quad\quad (\pm\sqrt{2}\omega_1, \pm\sqrt{2}\omega_2, \pm\sqrt{2}\omega_3)\end{aligned} \tag{26}$$

Cell structure of the $orbit - (1000) = V_0$**(16-cell)**
The orbit V_0, as we noted in (25), consists of two tetrahedra when projected into 3D under the group $W(SU(4))$. One can check that any set of four quaternions of V_0 orthogonal to each other form the vertices of a tetrahedron. The decomposition in (25) actually consists of the sets

$$V_0 = \{(1, e_1, e_2, e_3); (-1, -e_1, -e_2, -e_3)\} \tag{27}$$

each set representing a tetrahedron under the group $W(SU(4))$ given by the elements in (16). The generators f and g of $W(SU(4))$ defined in (15) permute the quaternionic units in the form:

$$\begin{aligned} f &: 1 \to e_1 \to e_2 \to e_3 \to 1 \\ g &: e_1 \to e_2 \to e_3 \to e_1 \end{aligned} \tag{28}$$

which leave the centers of two tetrahedra, $\pm\frac{\omega_0}{2}$, invariant as shown in (16). The index of $W(SU(4))$ in $W(SO(9))$ is 16, corresponding to the number of conjugate groups fixing one of the elements of the 8-cell of (24) which can also be written as

$$V_+ \oplus V_- = \{\pm\omega_a, \pm\bar{\omega}_a\},\ a = 0, 1, 2, 3. \tag{29}$$

For each choice of tetrahedral vertices there corresponds one $\pm\omega_a$ or $\pm\bar{\omega}_a$. For example, for a tetrahedron with the vertices of $1, -e_1, -e_2, -e_3$ the center left invariant is $\frac{\bar{\omega}_0}{2}$. Therefore, for 16 choice of tetrahedra out of V_0, we have a representation of $W(SU(4))$ in the form

$$W(SU(4)) = \{[p, \bar{\lambda}\bar{p}\lambda] \oplus [t, \lambda\bar{t}\lambda]^*\},\ p \in T,\ t \in T' \tag{30}$$

where λ is one of those quaternions in (29). Thus we have shown that the set of quaternions of V_0 form 16 tetrahedra, 8 of which meet at one vertex, implying that every vertex is connected to the nearest other 6 vertices forming 24 edges and 32 equilateral triangular faces. This polytope is denoted by $\{3, 3, 4\}$ in the Schlafli symbol and its dual is the polytope made by the centers of the tetrahedra which can be taken as the set of quaternions in (29). The hypercube $V_+ \oplus V_- = \{\pm\omega_a, \pm\bar{\omega}_a\}, a = 0, 1, 2, 3$ is represented by the Schlafli symbol $\{4, 3, 3\}$.

Cell structure of the(8-cell)
We have already seen in (24) that it is decomposed into two cubes under the group $W(SO(7)) \approx S_4 \times Z_2 = \{[p, \bar{p}] \oplus [t, \bar{t}] \oplus [p, \bar{p}]^* \oplus [t, \bar{t}]^*\}$ which leaves the quaternion ± 1 invariant. The index of $W(SO(7))$ in $W(SO(9))$ is 8, which corresponds to the number of vertices of V_0. This indicates that the conjugates of $W(SO(7))$ in $W(SO(9))$can be represented as

$$\begin{aligned} W(SO(7)) &\approx S_4 \times Z_2 \\ &= \{[p, \bar{\lambda}\bar{p}\lambda] \oplus [t, \bar{\lambda}\bar{t}\lambda] \oplus [p, \lambda\bar{p}\lambda]^* \oplus [t, \lambda\bar{t}\lambda]^*\} \end{aligned} \tag{31}$$

with $\lambda \in V_0$. Therefore the $orbit - (0001) = V_+ \oplus V_-$ consists of 8 cubes whose centers are given by the quaternions V_0 up to a scale factor. These two dual polytopes with the $W(SO(9))$ symmetry are regular and the rest of the polytopes represented by the vectors $\Lambda = (a_1a_2a_3a_4)$, $(a_i = 0, 1; i = 1, 2, 3, 4)$ are the semi-regular polytopes which are the analogous structures of the Archimedean polyhedra in 4D.

5. Cell structures of the $W(SO(9))$ orbits with $D_3 \times Z_2$ and $D_4 \times Z_2$ groups

The $W(SO(9))$ orbits with different sizes are made of different cells whose symmetries are the Coxeter-Weyl groups $W(SO(7)), W(SU(4)), D_3 \times Z_2$ and the group $D_4 \times Z_2$. Details of the cells possessing these symmetries will be discussed for each orbit, however, a general discussion regarding the prismatic groups $D_3 \times Z_2$ and $D_4 \times Z_2$ will be needed since they do not occur in every orbit of $W(SO(9))$. Similar to what we have in 3D where faces of Platonic or Archimedean polyhedra meet at one vertex, the cells in a given 4D polytope meet at one vertex. In what follows we discuss the structures of the prismatic cells with their symmetries and find out in which orbits they appear.

Cells of the prismatic group $D_3 \times Z_2$
The generators of the prismatic group $D_3 \times Z_2$ are obtained from (9) and act on the quaternionic units as follows

$$\begin{aligned} r_1r_2 &: 1 \to e_1 \to e_2 \to 1, \\ r_4 &: 1 \to 1, e_1 \to e_1, e_2 \to e_2, e_3 \to -e_3 \end{aligned} \quad (32)$$

which leave the unit vector $\frac{1}{\sqrt{3}}(1 + e_1 + e_2)$ invariant. Let us apply the action of these group elements on the highest weight $\Lambda = a+be_1+ce_2+de_3$ to obtain the six quaternions

$$\begin{aligned} &a + be_1 + ce_2 + de_3,\ ae_1 + be_2 + c + de_3, \\ &ae_2 + b + ce_1 + de_3,\ a + be_1 + ce_2 - de_3, \\ &ae_1 + be_2 + c - de_3,\ ae_2 + b + ce_1 - de_3. \end{aligned} \quad (33)$$

The quaternions of the upper line and the lower line define two sets of parallel equilateral triangles provided Dynkin indices take some special integer values. The vertices of the triangles in the upper and lower planes are connected to each other with the lines parallel to the quaternion $2de_3$. The edges of the triangles are represented by the quaternions

$$\begin{aligned} &(a - c) + (b - a)e_1 + (c - b)e_2 \\ &\quad = (a_1 + a_2) - a_1e_1 - a_2e_2 \\ &(a - c)e_1 + (b - a)e_2 + (c - b) \\ &\quad = (a_1 + a_2)e_1 - a_1e_2 - a_2 \\ &(a - c)e_2 + (b - a) + (c - b)e_1 \\ &\quad = (a_1 + a_2)e_2 - a_1 - a_2e_1. \end{aligned} \quad (34)$$

In order these lines represent the sides of an equilateral triangle the Dynkin indices a_1 and a_2 should satisfy the relation

$$a_1^2 + a_2^2 + a_1a_2 = 1. \quad (35)$$

One has the solutions, either $a_1 = 1,\ a_2 = 0$ or $a_2 = 1,\ a_1 = 0$. With these values the triangles have the edge length $\sqrt{2}$. Triangular prism with equal edge lengths requires that $d = \frac{1}{\sqrt{2}}$ leading to $a_4 = 1$. This analysis proves that the triangular prisms occur as cells in the $W(SO(9))$ orbits with highest weights $(10a_31)$ or $(01a_31)$. If both $a_1 = 1,\ a_2 = 1$ are taken, where (35) does not apply, we obtain two parallel hexagons with the sides represented by the quaternions $1 - e_1,\ 1 - e_2,\ e_1 - e_2,\ e_1 - 1,\ e_2 - 1,\ e_2 - e_1$. In this case, one has a hexagonal prismatic cell with the same symmetry group $D_3 \times Z_2$. Therefore, the orbits with the highest weight $(11a_31)$ will possess *hexagonal prisms* as cells.

Cells of the prismatic group $D_4 \times Z_2$
The generators $(r_1,\ r_2,\ r_3)$ of the prismatic group $D_4 \times Z_2$ are given by (9) and act on the quaternionic units as follows

$$\begin{aligned} r_3r_4 &: 1 \to 1,\ e_1 \to e_1,\ e_2 \to e_3,\ e_3 \to -e_2 \\ r_1 &: 1 \to e_1,\ e_1 \to 1,\ e_2 \to e_2,\ e_3 \to e_3 \end{aligned} \quad (36)$$

which leave the quaternion $\frac{1}{\sqrt{2}}(1 + e_1) \in T'$ invariant. The group acts on the highest weight $\Lambda = a+be_1+ce_2+de_3$ to produce 8 vertices of the square prism (cube)

$$\begin{aligned} &a + be_1 + ce_2 + de_3,\ a + be_1 + ce_3 - de_2, \\ &a + be_1 - ce_2 - de_3,\ a + be_1 - ce_3 + de_2, \\ &ae_1 + b + ce_2 + de_3,\ ae_1 + b + ce_3 - de_2, \\ &ae_1 + b - ce_2 - de_3,\ ae_1 + b - ce_3 + de_2. \end{aligned} \quad (37)$$

The quaternions in the upper two lines and lower two lines are representing the parallel squares provided either $a_3 = 1,\ a_4 = 0$ or $a_4 = 1,\ a_3 = 0$. The prism is a cube if $a_1 = 1$. This proves that we have cubic cells with $D_4 \times Z_2$ symmetry in the orbits either given by $(1a_210)$ or $(1a_201)$. If both $a_3 = a_4 = 1$ then we obtain an octagonal prism with the sides of octagons represented by the quaternions .

$$\begin{aligned} &e_2 - e_3, \sqrt{2}e_2, e_2 + e_3, \sqrt{2}e_3, \\ &\quad -e_2 + e_3, -\sqrt{2}e_2, -e_2 - e_3, -\sqrt{2}e_3. \end{aligned} \quad (38)$$

The orbit with the highest weight ($1a_211$) will have an octagonal prism cell with the symmetry $D_4 \times Z_2$. Regular polytopes being represented by the highest weights (1000) and (0001) do not possess the prismatic cells as we discussed in Section 4. In the next section we will discuss semi-regular polytopes, some of which have also prismatic cells.

6. Semi-regular polytopes with $W(SO(9))$ symmetry

The vertices of the semi-regular polytopes can be obtained from (21) by substituting the values of the Dynkin labels. We will discuss the structures of all semi-regular polytopes in turn.

The orbit-(0100) (rectified 16-cell)
It is the rectified 16-cell and has 24 vertices given by the quaternions

$$\pm 1 \pm e_1,\ \pm 1 \pm e_2,\ \pm 1 \pm e_3,\ \pm e_1 \pm e_2,\ \pm e_2 \pm e_3,\ \pm e_3 \pm e_1. \qquad (39)$$

They represent the vertices of 24-cell which has a larger symmetry $W(F_4)$. When we project it in 3D under the group $W(SO(7))$ the vertices $1 \pm e_1,\ 1 \pm e_2,\ 1 \pm e_3$ and $-1 \pm e_1, -1 \pm e_2, -1 \pm e_3$ represent the two copies of the octahedra and the remaining quaternions $\pm e_1 \pm e_2,\ \pm e_2 \pm e_3,\ \pm e_3 \pm e_1$ are the vertices of a cuboctahedron. Under the decomposition of $W(SU(4))$ we use the bases vectors $\omega_a,\ a = 0, 1, 2, 3$ and the set of quaternions in (39) now decompose as two octahedra with the vertices $\omega_0 \pm \omega_1,\ \omega_0 \pm \omega_2,\ \omega_0 \pm \omega_3$ and $-\omega_0 \pm \omega_1,\ -\omega_0 \pm \omega_2,\ -\omega_0 \pm \omega_3$ and the cuboctahedron with the vertices $\pm\omega_1 \pm \omega_2,\ \pm \omega_2 \pm \omega_3,\ \pm \omega_3 \pm \omega_1$.The set of quaternions in (39) form 8 octahedra possessing the $W(SO(7))$ symmetry and 16 octahedra with the $W(SU(4))$ symmetry. This polytope does not have prismatic cells.

The orbit-(0010) (rectified 8-cell)
This orbit does not have prismatic cells. It is the rectified 8-cell with 32 vertices,

$$\pm 1 \pm e_1 \pm e_2,\ \pm 1 \pm e_2 \pm e_3,\ \pm 1 \pm e_3 \pm e_1,\ \pm e_1 \pm e_2 \pm e_3. \qquad (40)$$

Under the decomposition of $W(SO(7))$ the vertices represent two copies of cuboctahedra and one cube. When the quaternions in (40) are expressed in terms of $\omega_a,\ a = 0, 1, 2, 3$ then they decompose as two tetrahedra and two truncated tetrahedra under $W(SU(4))$. To understand the cell structures of the orbit-(0010) we note that it is made of 8 cuboctahedra with the $W(SO(7))$ symmetry and 16 tetrahedra with $W(SU(4))$ symmetry. The cuboctahedron with the vertices $1 \pm e_1 \pm e_2,\ 1 \pm e_2 \pm e_3,\ 1 \pm e_3 \pm e_1$ possesses the $W(SO(7))$ symmetry leaving the quaternion 1 invariant. The other cuboctahedra can be analyzed similarly, for instance, $e_1 \pm e_2 \pm 1,\ e_1 \pm e_3 \pm 1,\ e_1 \pm e_2 \pm e_3$ is a cuboctahedron in the 3D space orthogonal to the quaternion e_1. Readers may verify that quaternions $1+e_1+e_2,\ 1+e_2+e_3,\ 1+e_3+e_1,\ e_1+e_2+e_3$ form a tetrahedron under the group $W(SU(4))$ leaving the vector ω_0 invariant. It is clear that (40) consists of 16 tetrahedra similar to what we have discussed above. Therefore the orbit-(0010) consists of 24 cells with 8 cuboctahedra and 16 tetrahedra.

The orbit-(1100)(truncated 16-cell)
When $a_1 = a_2 = 1, a_3 = a_4 = 0$ are substituted in (21) we obtain 48 vertices of the truncated 16-cell. Truncation creates 6 new vertices at each vertex of the 16-cell resulting in overall 48 vertices. They are given by the set of quaternions

$$\begin{aligned}&\pm 2 \pm e_i, \pm 1 \pm 2e_i,\ i = 1, 2, 3\\ &\pm 2e_1 \pm e_2, \pm 2e_2 \pm e_3, \pm 2e_3 \pm e_1\\ &\pm e_1 \pm 2e_2, \pm e_2 \pm 2e_3, \pm e_3 \pm 2e_1.\end{aligned} \qquad (41)$$

When we decompose them under $W(SO(7))$, we obtain four octahedra and one truncated octahedron with 24 vertices. A similar analysis can be carried under $W(SU(4))$. We will now discuss the cell structure. The vertices in (41) can be classified in 8 sets of octahedra, each of which is invariant under one conjugate group of $W(SO(7))$, $(\pm 2 \pm e_i, i = 1, 2, 3)$, $(\pm 2e_1 \pm e_2, \pm 2e_1 \pm e_3, \pm 2e_1 \pm 1)$, $(\pm 2e_2 \pm e_1, \pm 2e_2 \pm e_3, \pm 2e_2 \pm 1)$, $(\pm 2e_3 \pm e_1, \pm 2e_3 \pm e_2, \pm 2e_3 \pm 1)$. Some vertices form truncated tetrahedron possessing the symmetry $W(SU(4))$. For example, the vertices $(1+2e_1,\ 1+2e_2,\ 1+2e_3,\ 2+e_1,\ 2+e_2,\ 2+e_3,\ e_1+2e_2,\ e_2+2e_3,\ e_3+2e_1,\ 2e_1+e_2,\ 2e_2+e_3,\ 2e_3+e_1)$ constitute a truncated tetrahedron. Since the center of the above truncated tetrahedron is equal to ω_0 up to a scale factor it is invariant under the group $W(SU(4))$ given in (16). It is clear that we can construct the vertices of the truncated tetrahedron 16 different ways. Therefore, the orbit-(1100) has 24 cells, 8 of which are octahedra and the 16 are the truncated tetrahedra.

The orbit-(1010) (cantellated 16-cell)
Substitution of $a_1 = a_3 = 1,\ a_2 = a_4 = 0$ in (21)

results in 96 vertices which can be constructed from the set of quaternions $\pm 2 \pm e_1 \pm e_2$ by performing first cyclic permutations of e_1, e_2, e_3 and then the cyclic permutations of the units $1, e_1, e_2, e_3$.The first permutation yields to 24 vertices and the second permutations result in $24 \times 4 = 96$ vertices. Now we analyze its cell structure. The vertices

$$\pm 2 \pm e_1 \pm e_2, \ \pm 2 \pm e_2 \pm e_3, \ \pm 2 \pm e_3 \pm e_1 \quad (42)$$

represent two copies of cuboctahedra possessing the symmetry $W(SO(7))$ which leaves the quaternion ± 1 invariant. Since the other vertices of orbit-(1010) are obtained by permutations of $1, e_1, e_2, e_3$ we obtain altogether 8 cuboctahedra left invariant under the conjugate groups of $W(SO(7))$. The set of 12 vertices

$$\begin{aligned}
&2 + e_1 + e_2, \ 2 + e_2 + e_3, \ 2 + e_3 + e_1, \\
&2e_1 + e_2 + e_3, \ 2e_2 + e_3 + e_1, \ 2e_3 + e_1 + e_2, \\
&\text{and} \\
&2e_2 + e_3 + 1, \ 2e_3 + e_1 + 1, \ 2e_1 + e_2 + 1, \\
&2e_3 + 1 + e_1, \ 2e_1 + 1 + e_2, \ 2e_2 + 1 + e_3 \quad (43)
\end{aligned}$$

form another cuboctahedron under the symmetry $W(SU(4))$ in the hyperplane orthogonal to ω_0. The vertices in (43) can be written as

$$\begin{aligned}
&2\omega_0 + (\pm\omega_1 \pm \omega_2), \ 2\omega_0 + (\pm\omega_2 \pm \omega_3), \\
&2\omega_0 + (\pm\omega_3 \pm \omega_1) \quad (44)
\end{aligned}$$

proving that they are the vertices of a cuboctahedron in 3D orthogonal to ω_0. Since we have 16 conjugate subgroups $W(SU(4))$ of $W(SO(9))$ we obtain 16 cuboctahedra similar to (43) or (44). As we discussed in Section 5, this orbit has 24 cubic cells having prismatic symmetry $D_4 \times Z_2$ of order 16. This prismatic group leaves the quaternion $\frac{1}{\sqrt{2}}(1 + e_1) \in T'$ invariant and generates 8 vertices from the highest weight $1 + 2e_1 + e_2$ as follows:

$$\begin{aligned}
&1 + 2e_1 + e_2, \ 1 + 2e_1 + e_3, \ 1 + 2e_1 - e_2, \\
&1 + 2e_1 - e_3, e_1 + 2 + e_2, \ e_1 + 2 + e_3, \\
&e_1 + 2 - e_2, \ e_1 + 2 - e_3. \quad (45)
\end{aligned}$$

As we have proven for the general case, they are the vertices of a cube having the symmetry $D_4 \times Z_2$. Since we can embed $D_4 \times Z_2$ in $W(SO(9))$ in 24 different ways, we can construct 24 such cubes out of the vertices of the orbit-(1010). Therefore the number of cells of orbit-(1010) is $48 = 8 + 16 + 24$.

The orbit-(1001)(runcinated 8-cell also runcinated 16-cell)

The 64 vertices of the orbit can be written as $\pm\gamma + \delta(\pm e_1 \pm e_2 \pm e_3)+$ cyclic permutations of $(1, e_1, e_2, e_3)$ with

$$\gamma = 1 + \frac{1}{\sqrt{2}}, \ \delta = \frac{1}{\sqrt{2}}. \quad (46)$$

It has four types of cells, *cubes* with $W(SO(7))$ symmetry, *tetrahedra* with $W(SU(4))$ symmetry, *cubes* with $D_4 \times Z_2$ symmetry and *triangular prisms* with $D_3 \times Z_2$ symmetry. It is clear that under the group $W(SO(7))$ the quaternions $\delta(\pm e_1 \pm e_2 \pm e_3)$ form a *cube* and one obtains 8 cubes as cells of the orbit-(1001) after the cyclic permutations of unit quaternions. The decomposition of the vertices under the group $W(SU(4))$ results in 4 vertices of the type

$$\begin{aligned}
&(\frac{1}{2} + \sqrt{2})\omega_0 + \frac{1}{2}(\pm\omega_1 \pm \omega_2 \pm \omega_3) \\
&\text{(even number of (-) sign).} \quad (47)
\end{aligned}$$

It is a *tetrahedron* and the rest of the vertices will be classified as *tetrahedra* under the 16 conjugate subgroups $W(SU(4))$.There are cubic cells which display only the prismatic symmetry $D_4 \times Z_2$. Starting with the vertex $\gamma + \delta(e_1 + e_2 + e_3)$ and by applying the group generators of $D_4 \times Z_2$, one can generate 8 vertices

$$\begin{aligned}
&\gamma + \delta(e_1 + e_2 + e_3), \ \gamma + \delta(e_1 - e_2 + e_3), \\
&\gamma + \delta(e_1 - e_2 - e_3), \ \gamma + \delta(1 + e_2 - e_3) \quad (48)
\end{aligned}$$

and

$$\begin{aligned}
&\gamma e_1 + \delta(1 + e_2 + e_3), \ \gamma e_1 + \delta(1 - e_2 + e_3), \\
&\gamma e_1 + \delta(1 - e_2 - e_3), \ \gamma e_1 + \delta(1 + e_2 - e_3). \quad (49)
\end{aligned}$$

The vertices in (48) and (49) represent two parallel square planes with the edge length $\sqrt{2}$.The edges connecting upper and lower vertices are parallel to the vector $(1 - e_1)$ with the edge length $\sqrt{2}$. Therefore the vertices in (48-49) constitute a cube with the prismatic symmetry $D_4 \times Z_2$ leaving $\frac{1}{\sqrt{2}}(1 + e_1) \in T'$ invariant. One can find 24 such cubes having the prismatic symmetry $D_4 \times Z_2$ leaving one of the elements of T' invariant. The generators of the prismatic group $D_3 \times Z_2$ acting on the highest weight $\gamma + \delta(e_1 + e_2 + e_3)$ generate six vertices of the *triangular prism* leaving the unit vector $\frac{1}{\sqrt{3}}(1 + e_1 + e_2)$ invariant are given by

$$\begin{aligned}
&\gamma + \delta(e_1 + e_2 + e_3), \ \gamma e_1 + \delta(1 + e_2 + e_3), \\
&\gamma e_2 + \delta(e_1 + 1 + e_3) \quad (50)
\end{aligned}$$

$$\gamma + \delta(e_1 + e_2 - e_3),\ \gamma e_1 + \delta(1 + e_2 - e_3),$$
$$\gamma e_2 + \delta(e_1 + 1 - e_3). \quad (51)$$

The vertices in (50) and (51) form triangular planes parallel to each other connected by the vectors parallel to $\sqrt{2}e_3$. The edges of the prism have all the same length $\sqrt{2}$. The group $D_3 \times Z_2$ can be embedded in $W(SO(9))$ in 32 different ways leaving one of the following quaternions invariant in each embedding

$$\frac{1}{3}(\pm 1 \pm e_1 \pm e_2),\ \frac{1}{3}(\pm e_1 \pm e_2 \pm e_3),$$
$$\frac{1}{3}(\pm e_2 \pm e_3 \pm 1),\ \frac{1}{3}(\pm e_3 \pm 1 \pm e_1). \quad (52)$$

The orbit-(1001) consists of 80=8+16+24+32 cells of four types having all edge lengths equal $\sqrt{2}$.

The orbit-(0110) (bitruncated 8-cell also bitruncated 16-cell)
The 96 vertices of the orbit-(0110) can be obtained from (21) by substituting $a_1 = a_4 = 0,\ a_2 = a_3 = 1$ resulting in

$$\pm 2 \pm 2e_1 \pm e_2,\ \ \pm 1 \pm 2e_1 \pm 2e_2, \pm 2 \pm e_1 \pm 1e_2$$
$$+(\text{cyclic permutations of } 1, e_1, e_2, e_3) \quad (53)$$

The set of 24 quaternions,

$$2 \pm 2e_1 \pm e_2,\ 2 \pm 2e_2 \pm e_3,\ 2 \pm 2e_3 \pm e_1,$$
$$2 \pm e_1 \pm 2e_2,\ 2 \pm e_2 \pm 2e_3,\ 2 \pm e_3 \pm 2e_1 \quad (54)$$

form a truncated octahedron. Now it is clear that under the group $W(SO(7))$ 96 vertices in (53) form 8 truncated octahedra. When the quaternions in (53) expressed in terms of ω_a and $\bar{\omega_a},\ a = 0, 1, 2, 3$ we note that the set of quaternions

$$\frac{5}{2}\omega_0 + \frac{1}{2}(\pm 3\omega_1 \pm \omega_2 \pm \omega_3),$$
$$\frac{5}{2}\omega_0 + \frac{1}{2}(\pm \omega_1 \pm 3\omega_2 \pm \omega_3),$$
$$\frac{5}{2}\omega_0 + \frac{1}{2}(\pm \omega_1 \pm \omega_2 \pm 3\omega_3),$$
$$(\text{even number of (-) sign}) \quad (55)$$

constitute the vertices of a truncated tetrahedron and we have 16 of those type. There does not exist any cell of either symmetry $D_3 \times Z_2$ and $D_4 \times Z_2$ in this polytope.

The orbit-(0101) (cantellated 8-cell)
This orbit has 96 vertices of the quaternions given below:

$$\gamma(\pm 1 \pm e_1) + \delta(\pm e_2 \pm e_3),\ \delta(\pm 1 \pm e_1)$$
$$+\gamma(\pm e_2 \pm e_3)$$
$$+(\text{cyclic permutations of } e_1, e_2, e_3) (56)$$

The set of quaternions $\gamma \pm \gamma e_1 + \delta(\pm e_2 \pm e_3)$+(cyclic permutations of e_1, e_2, e_3) will constitute a *small rhombicuboctahedron* with the symmetry $W(SO(7))$ and we have 8 of them. When (56) is expressed in terms of ω_0 we note that the 6 vertices

$$(1+\sqrt{2})\omega_0 \pm \omega_1,\ (1+\sqrt{2})\omega_0 \pm \omega_2,\ (1+\sqrt{2})\omega_0 \pm \omega_3, \quad (57)$$

form an octahedron having the symmetry $W(SU(4))$. Taking one of the vertex in (56), say $\gamma(1+e_1)+\delta(e_2+e_3)$, we can generate a triangular prism by the action of generators of $D_3 \times Z_2$. One of the 32 prisms would have the vertices

$$\gamma(1 + e_1) + \delta(e_2 \pm e_3),\ \gamma(e_1 + e_2) + \delta(1 \pm e_3),$$
$$\gamma(e_2 + 1) + \delta(e_1 \pm e_3). \quad (58)$$

The number of cells of the orbit-(0101) will be 56=8+16+32.

The orbit-(0011) (truncated 8-cell)
The 64 vertices of the orbit can be written compactly

$$(\pm\gamma \pm \gamma e_1 \pm \gamma e_2 \pm \delta e_3)$$
$$+(\text{cyclic permutations of } 1, e_1, e_2, e_3). (59)$$

This orbit does not involve any cell possessing the prismatic symmetries. In the space orthogonal to the unit quaternion ± 1 the vertices in (59) form a truncated cube of 24 vertices under the Coxeter-Weyl group $W(SO(7))$. We have 8 such truncated cubes as we have explained before. The vertices

$$(\frac{3}{2}+\frac{1}{\sqrt{2}})\omega_0+\frac{1}{2}(\pm\omega_1\pm\omega_2\pm\omega_3)\ (\text{odd (-) sign}) \quad (60)$$

form a tetrahedron under the group $W(SU(4))$ and we have 16 of this type.

The orbit-(1110) (cantitruncated 16-cell)
The 192 vertices of the orbit can be written as

follows:

$$\begin{aligned}
&\pm 3 \pm 2e_1 \pm e_2 \\
&\quad +(\text{cyclic permutations of } (1, e_1, e_2, e_3)), \\
&\pm 1 \pm 3e_1 \pm 2e_2 \\
&\quad +(\text{cyclic permutations of } (1, e_1, e_2, e_3)), \\
&\pm 2 \pm 1e_1 \pm 3e_2 \\
&\quad +(\text{cyclic permutations of } (1, e_1, e_2, e_3)), \\
&\pm 3 \pm e_1 \pm 2e_2 \\
&\quad +(\text{cyclic permutations of } (1, e_1, e_2, e_3)), \\
&\pm 2 \pm 3e_1 \pm e_2 \\
&\quad +(\text{cyclic permutations of } (1, e_1, e_2, e_3)), \\
&\pm 1 \pm 2e_1 \pm 3e_2 \\
&\quad +(\text{cyclic permutations of } (1, e_1, e_2, e_3)),
\end{aligned} \tag{61}$$

The set of 48 vertices $\pm 3 \pm 2e_1 \pm e_2, \ \pm 3 \pm e_1 \pm 2e_2$ +(cyclic permutations of (e_1, e_2, e_3)) form two copies of a *truncated octahedron* under the octahedral group $W(SO(7))$ and we have 8 such cells in this orbit. When (61) is expressed in terms of the quaternions ω_a and $\bar{\omega_a}, \ a = 0, 1, 2, 3$ we note that the set of vertices $\pm 3\omega_0 \pm 2\omega_1 \pm \omega_2, \ \pm 3\omega_0 \pm \omega_1 \pm 2\omega_2$ + (cyclic permutations of $\omega_1, \omega_2, \omega_3,$) form two copies of another set of *truncated octahedra* with a smaller symmetry of $W(SU(4))$. We have 16 such truncated octahedra in the spaces orthogonal to the vectors $\pm\omega_a, \pm\bar{\omega_a}, a = 0, 1, 2, 3$. Since $a_4 = 0$ this orbit does not contain triangular prisms however it has cubic cells with the symmetry $D_4 \times Z_2$. The vertices of the cube invariant under the symmetry fixing the quaternion $\frac{1}{\sqrt{2}}(1 + e_1) \in T'$ are given by

$$\begin{gathered}
3 + 2e_1 + e_2, \ 3 + 2e_1 + e_3, \ 3 + 2e_1 - e_2, \\
3 + 2e_1 - e_3, \ 3e_1 + 2 + e_2, \ 3e_1 + 2 + e_3, \\
3e_1 + 2 - e_2, \ 3e_1 + 2 - e_3.
\end{gathered} \tag{62}$$

Therefore the orbit-(1110) is made of 48=8+16+24 cells of 8 truncated octahedra with $W(SO(7))$, 16 cells of truncated octahedra with $W(SU(4))$ and 24 cubes with $D_4 \times Z_2$ symmetries respectively.

The orbit-(1101) (runcitruncated 16-cell)
We substitute $a_1 = a_2 = a_4 = 1, \ a_3 = 0$ in (21) and redefine the coefficients in(21) with their numerical values $a = 2 + \frac{1}{\sqrt{2}}, \ b = 1 + \frac{1}{\sqrt{2}}, \ c = d = \frac{1}{\sqrt{2}}$. Since $c = d$ the number of vertices in (21) with the numerical values reduces to 192. The $W(SO(7))$ cells are the *small rhombicuboctahedra*. The vertices of one of this type of cells are given by

$$\begin{gathered}
a \pm be_1 \pm ce_2 \pm ce_3, \ a \pm be_2 \pm ce_3 \pm ce_1, \\
a \pm be_3 \pm ce_1 \pm ce_2.
\end{gathered} \tag{63}$$

The same argument implies that we have 8 such *small rhombicuboctahedra*. The cell of $W(SU(4))$ in the space orthogonal to ω_0 when expressed with the quaternions $\pm\omega_a, \bar{\omega_a}, \ a = 0, 1, 2, 3$ is a truncated tetrahedron with the vertices expressed as follows:

$$\begin{aligned}
&(\frac{5}{2} + \sqrt{2})\omega_0 + \frac{1}{2}(\pm 3\omega_1 \pm \omega_2 \pm \omega_3), \\
&(\frac{5}{2} + \sqrt{2})\omega_0 + \frac{1}{2}(\pm \omega_1 \pm 3\omega_2 \pm \omega_3), \\
&(\frac{5}{2} + \sqrt{2})\omega_0 + \frac{1}{2}(\pm \omega_1 \pm \omega_2 \pm 3\omega_3), \\
&\qquad (\text{even number of (-) sign}).
\end{aligned} \tag{64}$$

In this orbit, in addition to the above cells we have cubic cells with the symmetry $D_4 \times Z_2$ and the hexagonal prismatic cells with symmetry $D_3 \times Z_2$. The vertices of the cube are given by the quaternions

$$\begin{gathered}
a + be_1 + ce_2 + ce_3, \ a + be_1 + ce_3 - ce_2, \\
a + be_1 - ce_2 - ce_3, \ a + be_1 - ce_3 + ce_2, \\
ae_1 + b + ce_2 + ce_3, \ ae_1 + b + ce_3 - ce_2, \\
ae_1 + b - ce_2 - ce_3, \ ae_1 + b - ce_3 + ce_2,
\end{gathered} \tag{65}$$

with the above values of a, b, c. We have 24 such cubes in this orbit. As we discussed in Section 5, this orbit possesses hexagonal prisms since $a_1 = a_2 = a_4 = 1$. The vertices will be given by

$$\begin{gathered}
a + be_1 + ce_2 \pm ce_3, \ b + ae_1 + ce_2 \pm ce_3, \\
c + ae_1 + be_2 \pm ce_3, \ c + be_1 + ae_2 \pm ce_3, \\
b + ce_1 + ae_2 \pm ce_3, \ a + ce_1 + be_2 \pm ce_3.
\end{gathered} \tag{66}$$

The vertices of the hexagon in the upper plane with (+sign) and the vertices of the hexagon in the lower plane with ((-) sign) are in the cyclic order.

The orbit-(1011) (runcitruncated 8-cell)
192 vertices of this orbit can be written

$$\begin{aligned}
&\pm a \pm be_1 \pm be_2 \pm ce_3, \ \pm a \pm be_2 \pm be_3 \pm ce_1, \\
&\quad \pm a \pm be_3 \pm be_1 \pm ce_2 \\
&\quad +(\text{cyclic permutations of } 1, e_1, e_2, e_3).
\end{aligned} \tag{67}$$

The 48 vertices in (67) represent two copies of a *truncated cube* in the space orthogonal to the unit quaternion ± 1. Therefore we have 8 copies of a *truncated cube* having the symmetry $W(SO(7))$. Expressing the vertices in (67) in terms of the unit vectors $\pm\omega_a, \pm\bar{\omega_a},\ a = 0,1,2,3$ it will allow us to identify the cell structure of the symmetry $W(SU(4))$. Indeed the vertices

$$\begin{aligned}
&(2+\sqrt{2})\omega_0 + (\pm\omega_1 \pm \omega_2),\\
&(2+\sqrt{2})\omega_0 + (\pm\omega_2 \pm \omega_3),\\
&(2+\sqrt{2})\omega_0 + (\pm\omega_3 \pm \omega_1)
\end{aligned} \quad (68)$$

represent a *cuboctahedron* in the space orthogonal to the unit quaternion ω_0. For each space orthogonal to the unit vectors $\pm\omega_a, \pm\bar{\omega_a},\ a = 0,1,2,3$ we have one cell of cuboctahedron, resulting in 16 altogether. We have 32 triangular prisms as cells here too since we have $a_1 = a_4 = 1,\ a_2 = 0$. The vertices of the *triangular prism* whose edges orthogonal to the unit quaternion $\frac{1}{\sqrt{3}}(1+e_1+e_2)$ are given by the quaternions

$$\begin{aligned}
&a + be_1 + be_2 \pm ce_3,\ ae_1 + be_2 + b \pm ce_3,\\
&ae_2 + b + be_1 \pm ce_3.
\end{aligned} \quad (69)$$

As we explained before we have 32 of this type of cells invariant under one of the conjugate groups $D_3 \times Z_2$. Here we have an octagonal prism with $D_4 \times Z_2$ symmetry since $a_1 = a_3 = a_4 = 1$. Using the group elements of $D_4 \times Z_2$ defined in (36) the 8 vertices of one of the octagon are given by in the cyclic order

$$\begin{aligned}
&a + be_1 + be_2 + ce_3,\ a + be_1 + ce_2 + be_3,\\
&a + be_1 - ce_2 + be_3,\ a + be_1 - be_2 + ce_3,\\
&a + be_1 - be_2 - ce_3,\ a + be_1 - ce_2 - be_3,\\
&a + be_1 + ce_2 - be_3,\ a + be_1 + be_2 - ce_3.
\end{aligned} \quad (70)$$

The vertices of the other octagon parallel to the one above are obtained from (70) by replacing the quaternionic units $1 \leftrightarrow e_1$. Following the previous arguments we note that there are 24 octagonal prismatic cells. The number of cells here equals 80=8+16+32+24.

The orbit-(0111) (cantitrucated 8-cell)
192 vertices of this orbit can be written in terms of quaternions as

$$\begin{aligned}
&\pm a \pm ae_1 \pm be_2 \pm de_3,\ \pm a \pm ae_2 \pm be_3 \pm de_1,\\
&\pm a \pm ae_3 \pm be_1 \pm de_2\\
&+(\text{cyclic permutation of } 1, e_1, e_2, e_3)
\end{aligned} \quad (71)$$

The vertices given below

$$\begin{aligned}
&\pm a \pm ae_1 \pm be_2 \pm de_3,\ \pm a \pm ae_2 \pm be_3 \pm de_1,\\
&\pm a \pm ae_3 \pm be_1 \pm de_2,\ \pm a \pm be_1 \pm ae_2 \pm de_3,\\
&\pm a \pm be_2 \pm ae_3 \pm de_1,\ \pm a \pm be_3 \pm ae_1 \pm de_2
\end{aligned} \quad (72)$$

represent two copies of a *great rhombicuboctahedron* invariant under the group $W(SO(7))$ and we have 8 of this type. The set of vertices invariant under the group $W(SU(4))$ are given by the quaternions

$$\begin{aligned}
&(\frac{5}{2}+\sqrt{2})\omega_0 + (\pm 3\omega_1 \pm \omega_2 \pm \omega_3),\\
&(\frac{5}{2}+\sqrt{2})\omega_0 + (\pm\omega_1 \pm 3\omega_2 \pm \omega_3),\\
&(\frac{5}{2}+\sqrt{2})\omega_0 + (\pm\omega_1 \pm \omega_2 \pm 3\omega_3)\\
&\text{(even number of (-) sign).}
\end{aligned} \quad (73)$$

They represent the cell of *truncated tetrahedron.* The vertices of the triangular prism invariant under the group $D_3 \times Z_2$ generated by the elements (32) are the following quaternions

$$\begin{aligned}
&a + ae_1 + be_2 \pm de_3,\ ae_1 + ae_2 + b \pm de_3,\\
&ae_2 + a + be_1 \pm de_3.
\end{aligned} \quad (74)$$

This orbit consists of 56 cells with 8 *great rhombicuboctahedra*, 16 *truncated tetrahedra* and 32 triangular prisms.

The orbit-(1111) (ommitruncated 8-cell also ommitruncated 16-cell)
This is the semi-regular orbit with largest size 384 and the vertices are obtained when $a_1 = a_2 = a_3 = a_4 = 1$ are substituted in (21) leading to the values

$$a = 3 + \frac{1}{\sqrt{2}},\ b = 2 + \frac{1}{\sqrt{2}},\ c = 1 + \frac{1}{\sqrt{2}},\ d = \frac{1}{\sqrt{2}}. \quad (75)$$

The 48 quaternions

$$\begin{aligned}
&a \pm be_1 \pm ce_2 \pm de_3,\ a \pm ce_1 \pm be_2 \pm de_3\\
&+(\text{cyclic permutations of } 1, e_1, e_2, e_3)
\end{aligned} \quad (76)$$

with the above values of b, c, d represent the vertices of a *great rhombicuboctahedron* in the space orthogonal to the unit quaternion ± 1. Similarly the following 24 quaternions

$$\begin{aligned}
&(3+\sqrt{2})\omega_0 \pm 2\omega_1 \pm \omega_2,\\
&(3+\sqrt{2})\omega_0 \pm \omega_1 \pm 2\omega_2\\
&+(\text{cyclic permutations of } \omega_1, \omega_2, \omega_3)
\end{aligned} \quad (77)$$

constitute the vertices of a *truncated octahedron*. The arguments we have discussed in Section 5 leads us to the conclusion that the orbit-(1111) has hexagonal and octagonal prismatic cells with the symmetries $D_3 \times Z_2$ and $D_4 \times Z_2$ respectively. The vertices of the *hexagonal prism* edges of which orthogonal to $\frac{1}{\sqrt{3}}(1 + e_1 + e_2)$ will be given by

$$\begin{aligned} &a + be_1 + ce_2 \pm de_3,\ b + ae_1 + ce_2 \pm de_3, \\ &ae_1 + be_2 + c \pm de_3,\ be_1 + ae_2 + c \pm de_3, \\ &ae_2 + b + ce_1 \pm de_3,\ be_2 + a + ce_1 \pm de_3. \end{aligned} \tag{78}$$

Finally the vertices of octagon in the upper plane of the *octagonal prism* are given by the quaternions

$$\begin{aligned} &a + be_1 + ce_2 + de_3,\ a + be_1 + de_2 + ce_3, \\ &a + be_1 + ce_3 - de_2,\ a + be_1 + de_3 - ce_2, \\ &a + be_1 - ce_2 - de_3,\ a + be_1 - de_2 - ce_3, \\ &a + be_1 - ce_3 + de_2,\ a + be_1 - de_3 + ce_2. \end{aligned} \tag{79}$$

The vertices of the octagon in the lower plane will be obtained from (79) by interchanging the quaternionic units $1 \leftrightarrow e_1$. Therefore, altogether the orbit-(1111) consists of 80=8+16+32+24 cells of *great rhombicuboctahedra*, *truncated octahedra*, *hexagonal prisms* and *octagonal prisms*. Names of the polyhedra are listed in Table II while a summary of the $W(SO(9))$ polytope decomposition can be found in Table III.

7. Concluding remarks

We have demonstrated an interesting correspondence between the orbits of the Coxeter-Weyl group $W(SO(9))$ obtained from highest weights $\Lambda = (a_1\ a_2\ a_3\ a_4)$ where the Dynkin labels take $a_i = 0$ or $a_i = 1$ only and the regular as well as semi-regular polytopes possessing the symmetry $W(SO(9))$. Moreover, we have used the quaternionic representation of the Coxeter-Weyl group $W(SO(9))$ so that the action of the group on the quaternionic representation of the highest weight $\Lambda = a + be_1 + ce_2 + de_3$ is simple to obtain the quaternionic vertices of the polytopes of interest. What $W(SO(9))$ does is just to change the signs and permute the quaternionic units $1, e_1, e_2, e_3$. We have explicitly studied the cell structures of the polytopes with the subgroups $W(SO(7))$, $W(SU(4))$, $D_4 \times Z_2$ and $D_3 \times Z_2$. Duality between the regular polytopes orbit-(1000) and orbit-(0001) is manifest where the quaternions fixed by the group $W(SU(4))$ are the vertices of the cells of $W(SO(7))$ and vice versa. Cell structures of all semi-regular polytopes have been explicitly worked out and the result is presented in Table III.

TABLE II: Platonic and Archimedean solids designated by Dynkin labels of $W(SO(9))$ and $W(SU(4))$ and prismatic polyhedra [2]. Snub cube and snub dodecahedron are excluded from the list as they require non-linear relations between the Dynkin labels.

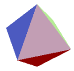
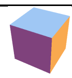
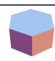
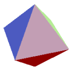
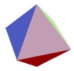
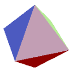
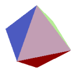
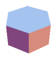
Symmetry	Order	Dynkin label	Name	Figure
$W(SU(4))$	24	(100) or (001)	Tetrahedron	
$W(SU(4))$	24	(110) or (011)	Truncated tetrahedron	
$W(SO(7))$	48	(100)	Octahedron	
$W(SO(7))$	48	(001)	Cube	
$W(SO(7))$	48	(010)	Cuboctahedron	
$W(SO(7))$	48	(110)	Truncated octahedron	
$W(SO(7))$	48	(011)	Truncated cube	
$W(SO(7))$	48	(101)	Small rhombicuboctahedron	
$W(SO(7))$	48	(111)	Great rhombicuboctahedron	
$D_3 \times Z_2$	12		Triangular prism	
$D_3 \times Z_2$	12		Hexagonal prism	
$D_4 \times Z_2$	16		Octagonal prism	

TABLE III: W(SO(9)) orbits as regular and semi-regular polytopes.

	Cell counts by symmetry				Element counts			
Dynkin label	$W(SO(7))$ cells (8)	$D_4 \times Z_2$ cells (24)	$D_3 \times Z_2$ cells (32)	$W(SU(4))$ cells (16)	Cells	Faces	Edges	Vertices
(1000) 16-cell					16	32	24	8
(0001) 8-cell					8	24	32	16
(0100)					24	96	96	24
(0010)					24	88	96	32
(1100)					24	96	120	48
(1010)					48	240	288	96
(1001)					80	208	192	64
(0101)					56	248	288	96
(0011)					24	88	128	64
(0110)					24	120	192	96

(cont.)

	Cell counts by symmetry				Element counts			
Dynkin label	$W(SO(7))$ cells (8)	$D_4 \times Z_2$ cells (24)	$D_3 \times Z_2$ cells (32)	$W(SU(4))$ cells (16)	Cells	Faces	Edges	Vertices
(1110)					48	240	384	192
(1101)					80	368	480	192
(1011)					80	368	480	192
(0111)					56	248	384	192
(1111)					80	464	768	384

References

[1] Michael Atiyah and Paul Sutcliffe, Milan Journal of Math. Vol 71,II 1/September 2003(arXiv:math-ph/0303071 v1 31 Mar 2003).

[2] M. Koca, M. Al-Ajmi and R. Koc, J. Math. Phys, **48**, 113514 (2007).

[3] B. Champagne, M. Kjiri, J. Patera and R. T. Sharp, Can. J. Phys. **73**, 566 (1995).

[4] H. W. Croto, J. R. Heath, S. C. OBrien and R. E. Smalley, Nature **318**, 162 (1985).

[5] T. S. Baker, N. H. Olson and S. D. Fuller, Microbiol. Mol. Biol. Rev. **63**, 862 (1999).

[6] M. Koca, J. Phys. Math. Gen. **A22**, 1949 (1989); J. Phys. Math. Gen. **A22**, 4125 (1989).

[7] T. Pengpan and P. Ramond, Phys. Rep. **C315**, 137 (1999); P. Ramond, "Algebraic Dreams, UFIFT-HET-01-27.

[8] For a review see for instance, Bobby S. Acharya and Sergei Gukov, Phys. Rep. **C392**, 141 (2004).

[9] M. Koca, R. Koc and M. Al-Barwani, J. Math. Phys. **44**, 3123 (2003); M. Koca, R. Koc and M. Al-Barwani, J. Math. Phys. **47**, 043507-1 (2006).

[10] H. S. M. Coxeter, *Regular Complex Polytopes* (Cambridge University Press, Cambridge, 1973).

[11] P. du Val, *Homographies, Quaternions and Rotations* (Oxford University Press, 1964).

[12] http://en.wikipedia.org/wiki/Uniform_polychoron; http://members.aol.com/Polycell/Section2.html.

[13] J. McKay, Am. Math. Soc. Proc. Sym. Pure Math. **37** 183 (1980).

[14] M. Koca, R. Koc and M. Al-Barwani, J. Phys. A: Math. Gen. **34**, 11201 (2001); J. H. Conway and D. A. Smith, *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry* (A. K. Peters, Ltd., 2003); M. Koca, R. Koc, M. Al-Barwani and S. Al-Farsi, Linear Algeb. Appl. **412**, 441 (2006).

[15] W. G. McKay and J. Patera, *Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras* (Dekker, New York, 1981; R. Slansky, Phys. Rep. **79**, 1 (1981).

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